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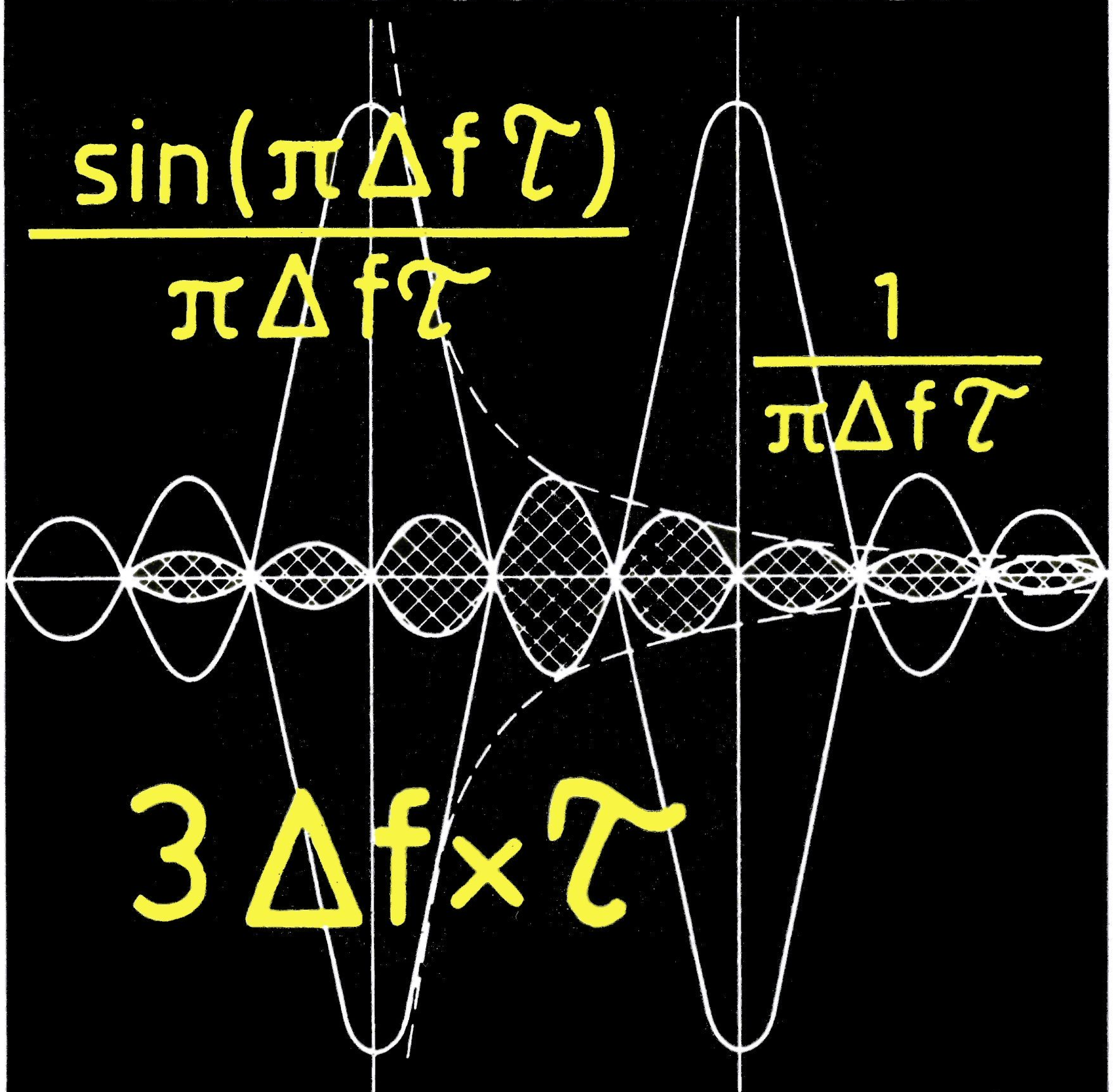
CROSS-CORRELATION



AND CROSS-SPECTRUM ANALYSES

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On the Applicability and Limitations of the Cross-Correlation and the Cross-Spectral Density Techniques

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ABSTRACT

As the cross-correlation function and the cross-spectral density function are directly related via Fourier transforms they contain the same amount of information. In practice, however, one representation may, under certain circumstances, be preferred to the other. It is shown that when the transmission paths within a system is frequency independent the correlation function technique may render results which are more readily interpreted than results obtained from cross-spectral density measurements. On the other hand, when the system transmission paths are frequency dependent (and practically independent of time) the cross-spectral density representation is superior to the cross-correlation function representation. Time and frequency limitations imposed on the two types of information representation are discussed and analytical expressions for the limitations formulated. The analytical expressions have been tested experimentally, partly by direct analog measurements and partly by digital processing of analog data.

ZUSAMMENFASSUNG

Da die Kreuzkorrelationsfunktion und die Kreuzleistungsdichtefunktion sich mittels Fouriertransformation ineinander überführen lassen, haben sie den gleichen Informationsgehalt. In der Praxis wird jedoch u. U. eine Darstellung der anderen vorgezogen. Hier wird gezeigt, daß die Korrelationsfunktion leichter interpretierbare Resultate liefern kann als die Leistungsdichtefunktion, wenn die Übertragungswege des untersuchten Systems von der Frequenz unabhängig sind. Andererseits ist die Kreuzleistungsdichte-Darstellung der Kreuzkorrelations-Darstellung überlegen, wenn die Übertragungswege frequenzabhängig (und praktisch zeitunabhängig) sind.

Die zeitlichen und frequenzmäßigen Grenzen beider Darstellungsarten werden diskutiert und in Gleichungen formuliert. Diese analytisch gewonnenen Ausdrücke wurden experimentell nachgeprüft, teils durch direkte Analogmessungen – teils durch digitale Verarbeitung analoger Meßdaten.

SOMMAIRE

La fonction d'intercorrélation et la fonction de densité interspectrale sont liés directement par la transformée de Fourier. Elles contiement donc la même somme d'information. En pratique, cependant, une représentation peut être preferée a l'autre. On montre, que, pour un système linéaire en fréquence, les résultats obtenus par l'utilisation de la fonction d'intercorrélation sont plus facile à interpréter et que, par contre, pour un système sélectif (et pratiquement independant du temps) la fonction de sensité interspectrale est mieux adaptée.

Les limites en temps et fréquence, pour les deux types d'informations, sont discutées et leurs expressions analytique sont données. Des mesures expérimentales ont été effectuées sur ces expression, analytiques, partiellement par des mesures analogiques directes, partiellement par un traitement digital des données analogiques.

Introduction

There are several methods of relating measurement data observed at a certain point in a system to data obtained at some other observation point within the same system. The most straight forward method may simply be to compare the data directly by eye.

However, even if the human eye is an amazingly sensitive and selective measuring device situations occur where such comparison is extremely difficult. Also, although a qualitative measure of the relationship between data may be obtained by merely looking at the time records, it is not possible, in general, to obtain a quantitative measure of the relationship by means of this "method".

Mathematical physicists have therefore introduced the so-called *cross-correlation* function, which actually gives a *quantitative measure* of the relationship

$$\psi_{xy}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_0^T f_x(t)\, f_y(t + \tau)\, dt$$

Here $f_x(t)$ is the magnitude of a signal observed at the point x at an arbitrary instant of time, t, and $f_y(t + \tau)$ is the magnitude of a signal observed at a point y a time τ later. By varying τ a complete function of the relationship between the signals at x and y as a function of time delays is obtained.

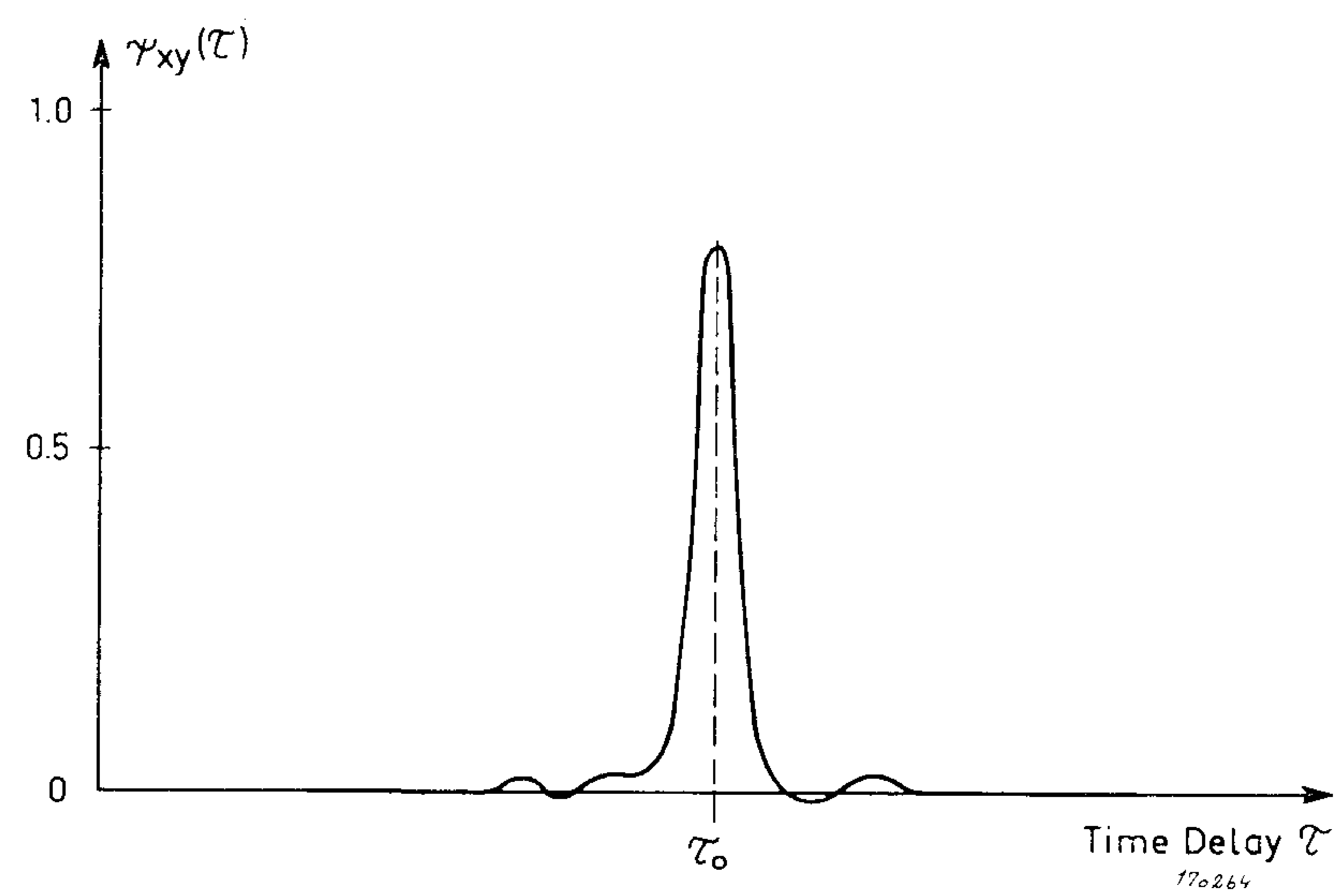


Fig. 1. Sketch indicating the cross-correlation function for a hypothetical frequency independent random process.

Fig. 1 shows a "classic" example of such a correlation function. It can be seen that when τ is zero practically no relationship exists between the two signals. By increasing τ it becomes evident, however, that a certain relationship does exist. At a certain delay-time, τ_0, this relationship reaches a

maximum, whereafter the relationship again decreases to zero for large values of τ.
A relationship of the kind indicated in Fig. 1 is typical for a system which is *frequency independent* and contains some sort of delay mechanism, for instance the transmission of sound in a non-absorptive medium. Such a system will be further discussed later in this paper.
If the maximum value of $\psi_{xy}(\tau)$ is equal to unity the signal at y is exactly the same signal as that at x, but delayed a time τ_o. On the other hand, if $\psi_{xy}(\tau_o)$ is less than unity only a certain part of the signal observed at x is present at y. The case illustrated in Fig. 1 is an idealized case and this kind of correlation function is found rather rarely in practice because normally the system within which observations are made is frequency dependent. To investigate the frequency dependency use may be made of the so-called *Fourier transform* method. The result of applying the Fourier transform to the correlation function is the *cross-spectral density function:*

$$W_{xy}(f) = \int_{-\infty}^{\infty} \psi_{xy}(\tau)\, e^{-j2\pi f\tau}\, d\tau$$

This function is, in general, complex, containing both real and imaginary terms, a fact which is readily seen in that both magnitude and phase measures should be preserved. (Time delays, for instance, represent phase differences in the frequency domain).
To represent the real part of the cross-spectral density function a function termed the co-spectrum, $C_{xy}(f)$, is used. Similarly, the imaginary part of the cross-spectral density function is represented by the quad- (quadrature) spectrum, $Q_{xy}(f)$.

Then:

$$W_{xy}(f) = C_{xy}(f) - j\, Q_{xy}(f)$$

or

$$|W_{xy}(f)| = \sqrt{C^2_{xy}(f) + Q^2_{xy}(f)}$$

and

$$\varphi_{xy}(f) = \tan^{-1}\left[\frac{Q_{xy}(f)}{C_{xy}(f)}\right]$$

where $|W_{xy}(f)|$ is the absolute magnitude modulus of the cross-spectral density function at the frequency f, and $\varphi_{xy}(f)$ is the phase difference between the signal at x and the signal at y.
Before finishing these introductory remarks on cross-correlation functions and cross-spectral densities it should be noted that if the cross-spectral density

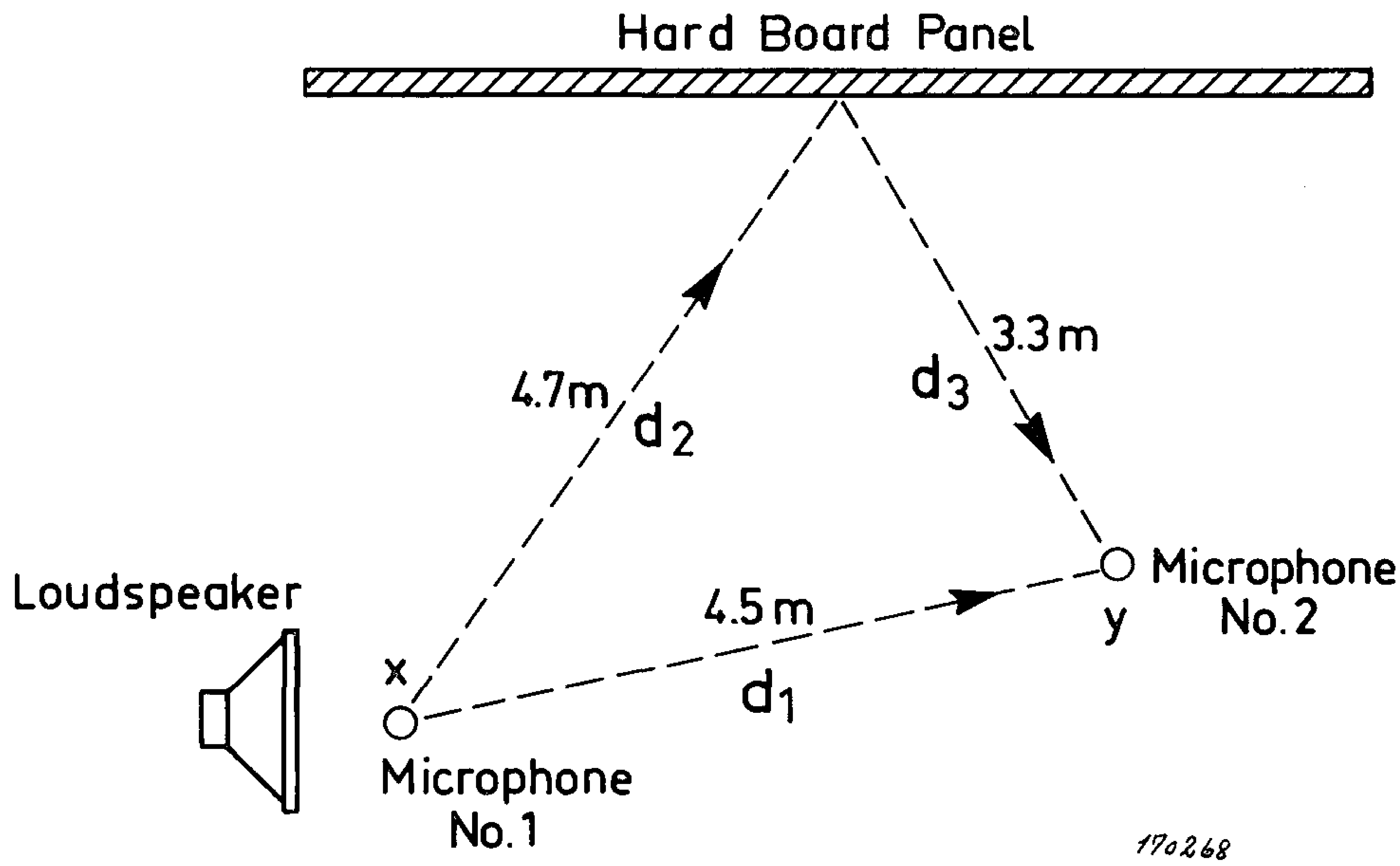


Fig. 3. Measuring arrangement used to investigate the limitations imposed upon the cross-correlation function techniques when both frequency and time dependent paths are present in a system.

resonant modes in the transmission path. If more resonant modes and/or reflections had been present the correlogram had become still more complicated.

To try and investigate the limitations imposed upon the cross-correlation function techniques when *both frequency and time dependent paths are* present in a system some simple experiments were carried out at Brüel & Kjær. The experiments consisted in measuring the correlation function between the signals obtained from two microphones placed in an acoustic field, Fig. 3.

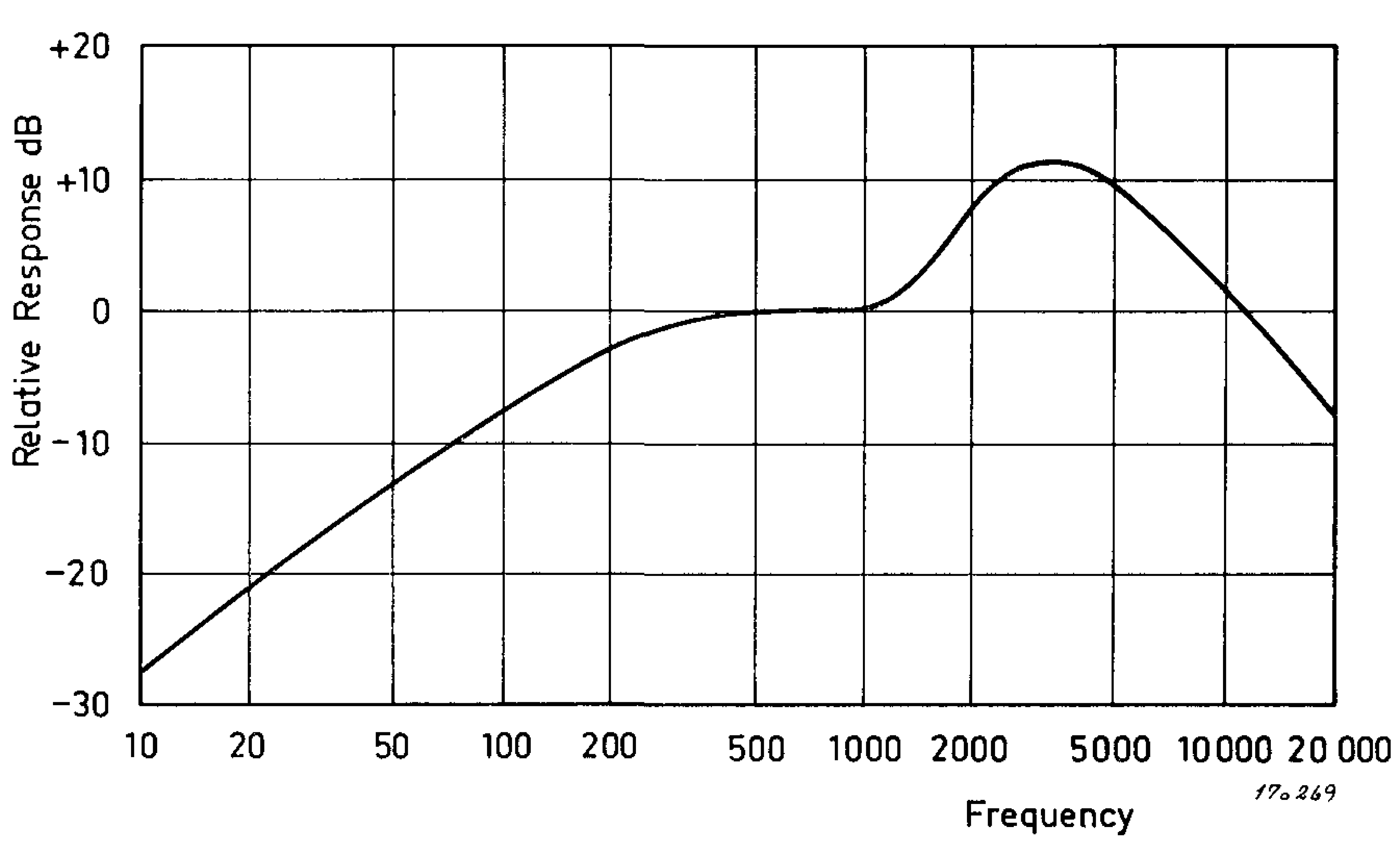


Fig. 4. The frequency weighting characteristic used to shape the wide band random noise feeding the loudspeaker.

The desired acoustic field was produced by transmitting filtered random noise via a loudspeaker into an anechoic chamber with one hard, reflecting "wall", see Fig. 3. (Actually, the reflecting "wall" was produced by installing a large hard-board panel in the anechoic room). The signal received by microphone no. 2 thus consisted of the direct sound from the loudspeaker plus *one reflection only.*

From the distances noted in the figure and the speed of sound in air the delay time, τ_1, for the direct sound path can be found:

$$\tau_1 = \frac{d_1}{c_o} = \frac{4.5}{344} = 0.013 \text{ s}$$

where $c_o = 344$ m/s = speed of sound.

Similarly, the delay time, τ_2, for the reflected sound is:

$$\tau_2 = \frac{d_2 + d_3}{c_o} = \frac{4.7 + 3.3}{344} = \frac{8}{344} \approx 0.0235 \text{ s}$$

The difference in delay between the two signals received by microphone no. 2 is thus:

$$\tau_s = \tau_2 - \tau_1 = 0.023 - 0.013 = 0.010 \text{ s} = 10 \text{ ms}$$

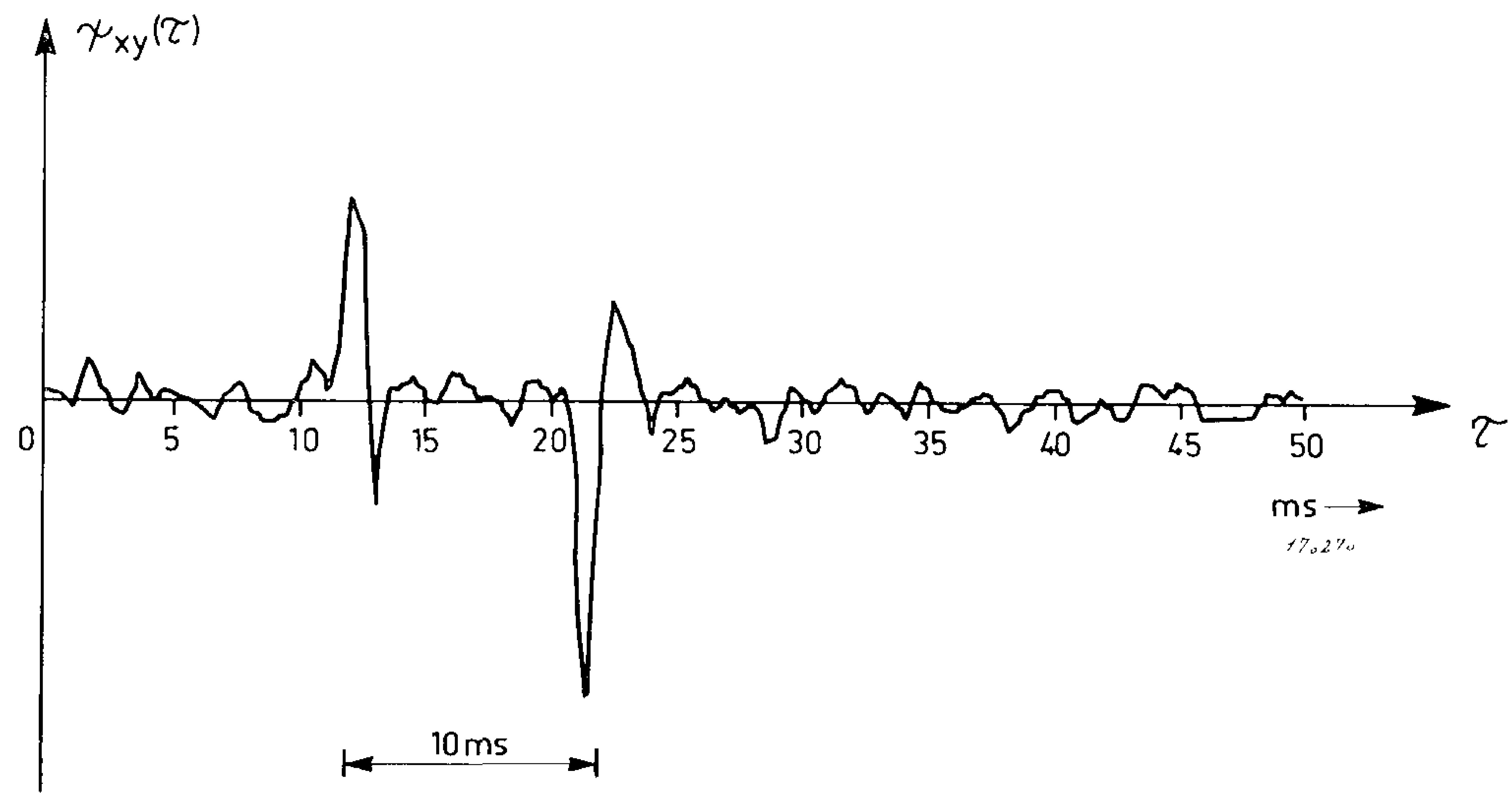


Fig. 5. Cross-correlation function (correlogram) obtained from measurements with shaped, wide band, random noise.

When the loudspeaker was fed with wide band random noise, frequency weighted according to the curve shown in Fig. 4 (D-weighting), a correlogram as given in Fig. 5 was obtained.

By now restricting the transmitted frequency band to 1/3 octave at 630 Hz, i.e. the bandwidth of the transmitted band being some 150 Hz, Fig. 6, a correlogram of the type shown in Fig. 7 was produced. Note that in this case it is somewhat difficult to separate the correlation "peak" produced by the reflected sound from that produced by the direct sound.

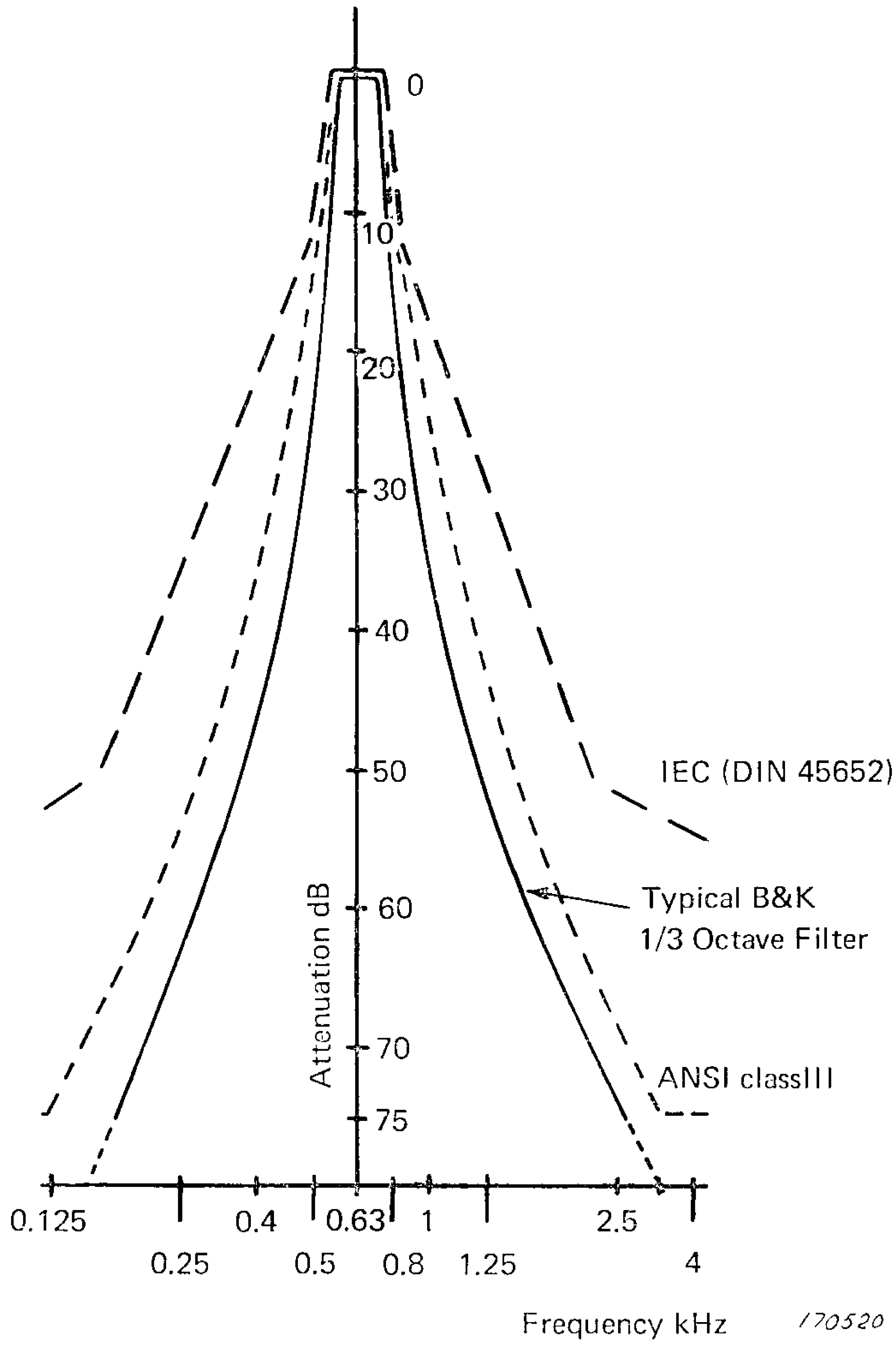


Fig. 6. Typical frequency characteristic of a 1/3 octave filter.

These experiments suggest that a limitation of the form

$$\Delta f_{min}\,\tau_s = f\,[\psi_{xy}(\tau)]$$

exists in correlation measurements, where Δf_{min} is the bandwidth of the signals to be correlated, and τ_s is the time delay between the events that can be properly separated.

The function $f\,[\psi_{xy}(\tau)]$ is further discussed in Appendix A, where it is shown that if a magnitude error of some 10 % in the absolute value of the maximum correlation is acceptable then

$$f\,[\psi_{xy}(\tau)] \approx 3\,\frac{\psi_{xy}(\tau_1)}{\psi_{xy}(\tau_2)}$$

Here $\psi_{xy}(\tau_1)\,/\,\psi_{xy}(\tau_2)$ represents the ratio between two successive correlation maxima, see Fig. 8.

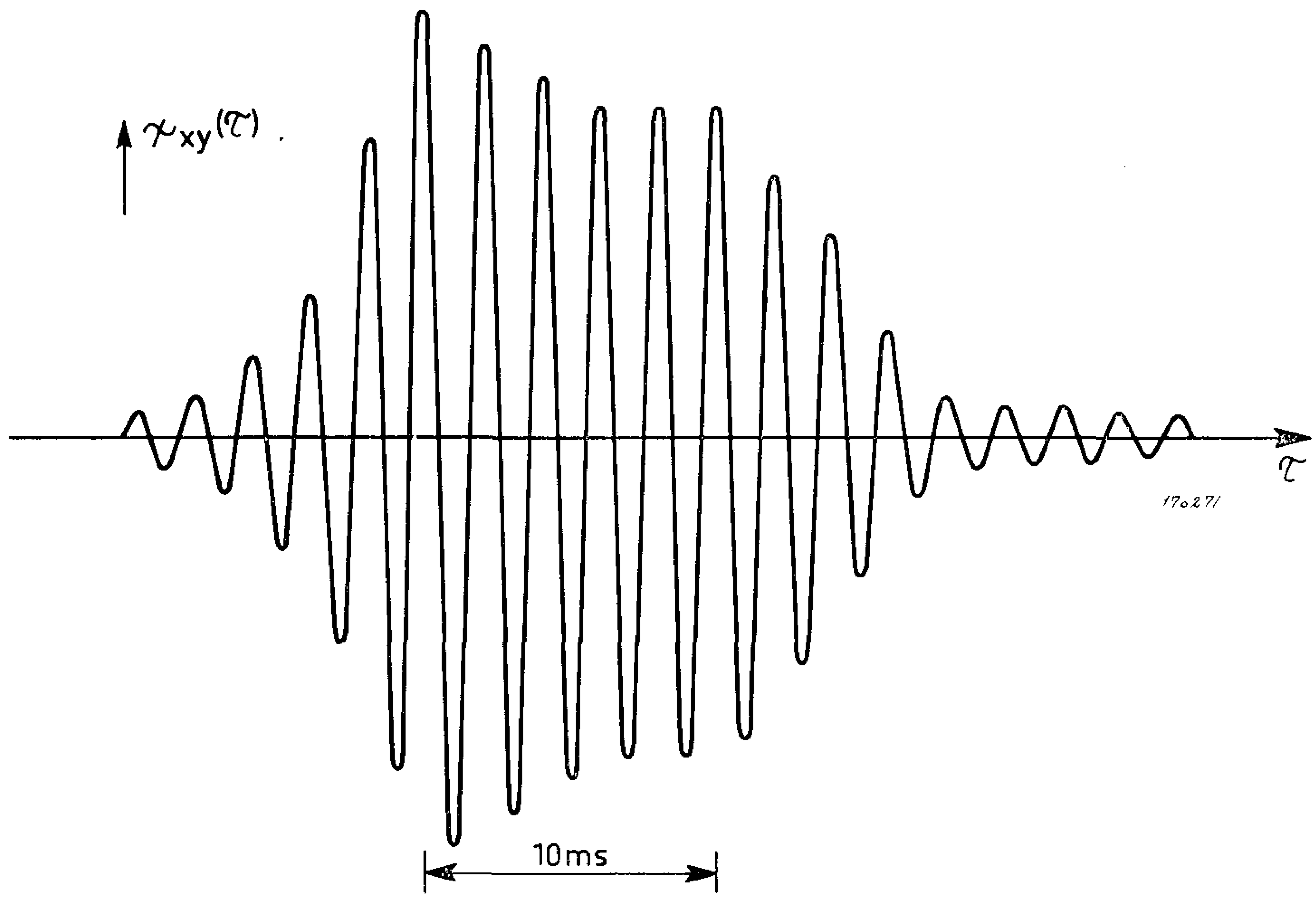


Fig. 7. Correlogram obtained from measurements using 1/3 octave band of noise centered at 630 Hz.

The formula

$$\Delta f_{min} \times \tau_s \geq 3 \frac{\psi_{xy}(\tau_1)}{\psi_{xy}(\tau_2)}$$

imposes a rather severe limitation upon the practical use of correlation functions in the fields of acoustics and vibration.

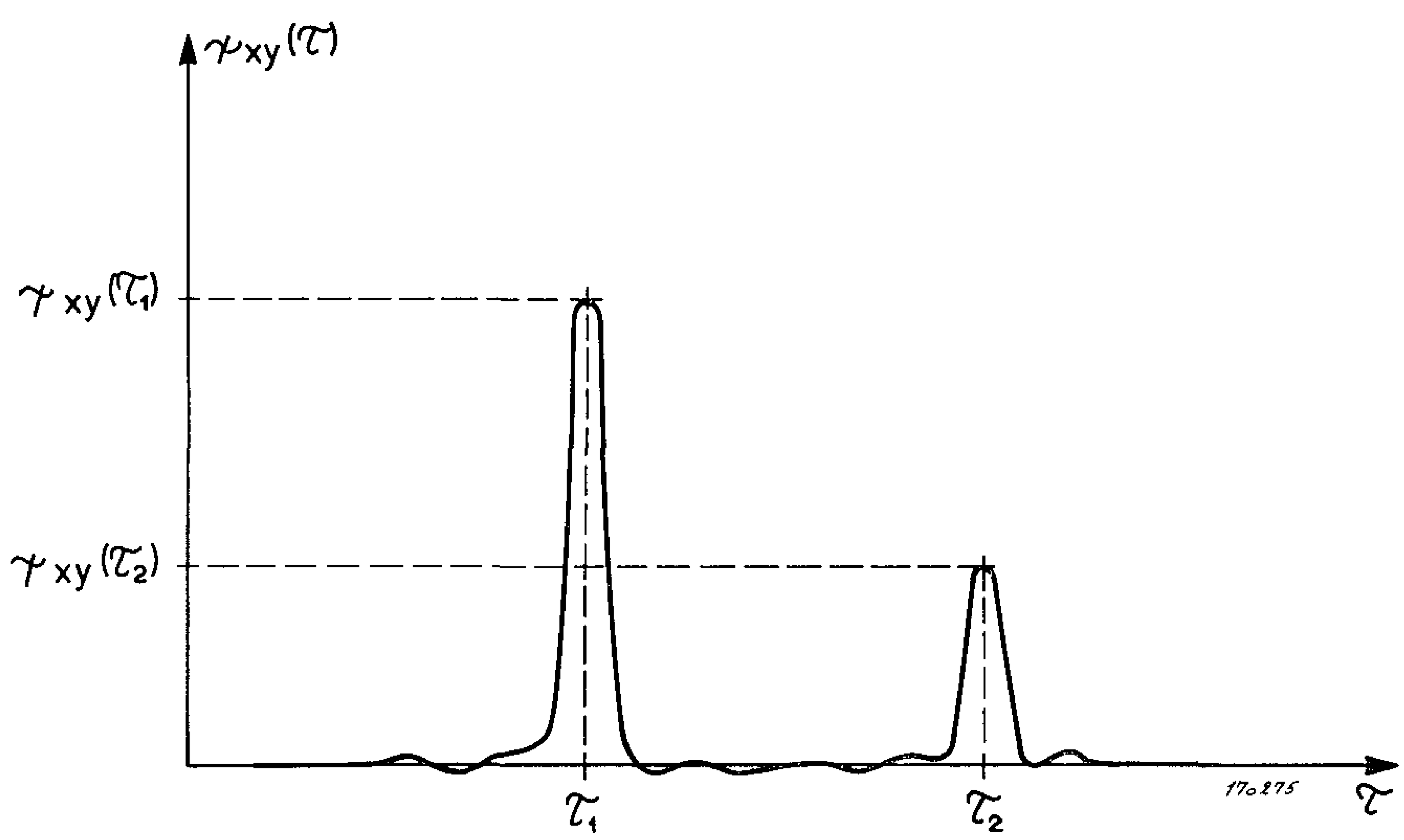


Fig. 8. Sketch illustrating cross-correlation maxima and time delays in a hypothetical random process.

In the field of acoustics, for instance, time delays of the order of 10 ms or less between reflections are not uncommon. Even if it is assumed that the magnitudes of the reflected signals are of equal strength ($\psi_{xy}(\tau_1) = \psi_{xy}(\tau_2)$) then

$$\Delta f_{min} \times \tau_s \geq 3 \text{ i.e. } \Delta f_{min} \geq 300 \text{ Hz}$$

This means that the *minimum* absolute bandwidth to be used in correlation measurements will here be of the order of 300 Hz. An optimum frequency resolution of this order of magnitude may in many cases not be satisfactory and it is therefore deemed that the use of correlation functions in acoustics is of very limited technical value. Also, most acoustic problems may be solved equally well by other, in general, simpler methods.

Turning now to the fields of structural mechanical vibrations where the time between reflections are in the region of 1 ms or less, and complicated, lightly damped (narrow band) resonances are present, correlation function techniques in the time domain seem a quite unrealistic proposition. Here, however, the cross-spectral density techniques seem to offer considerable possibilities. This is further discussed in the next section of the paper.

Applicability of the Cross-Spectral Density Techniques

It was stated in the introduction that the cross-spectral density function, $W_{xy}(f)$, is a complex function, consisting of real, $C_{xy}(f)$, as well as imaginary, $Q_{xy}(f)$, terms i.e.:

$$W_{xy}(f) = C_{xy}(f) - j\,Q_{xy}(f)$$

The actual determination of these terms can be made by means of a measuring arrangement as sketched in Fig. 9.

From theoretical considerations*) it can be shown that:

$$C_{xy}(f) = \lim_{\Delta f \to 0} \lim_{T \to \infty} \frac{1}{\Delta f\, T} \int_0^T f_{x\Delta f}(t) \quad f_{\Delta f}(t)\, dt$$

and

$$Q_{xy}(f) = \lim_{\Delta f \to 0} \lim_{T \to \infty} \frac{1}{\Delta f\, T} \int_0^T f_{x\Delta f}(t) \quad f^{*}_{\Delta f}(t)\, dt$$

where $f_{x\Delta f}(t)$ is the output signal from a filter with bandwidth Δf at the measurement point *x*, and $f_{y\Delta f}(t)$ is the output of an exactly equal filter at the measurement point *y*. $f^{*}_{y\Delta f}(t)$ is equal to $f_{\Delta f}(t)$ shifted 90° in phase. The products of the signals are averaged over a time *T*, see also Fig. 9.

*) See also: J. T. Broch: "On the Measurement and Interpretation of Cross-Power Spectra". Brüel & Kjær Techn. Rev. No. 3 — 1968.

From the figure it can be seen that the measurement of co- and quad spectral density functions is a rather straight forward matter. Also in this case, however, certain practical restrictions regarding filter bandwidths and signal time delays are imposed upon the measurements. While in the case of correlation time function measurements a limitation of the form

$$\Delta f_{min} \times \tau_s \geqq \text{const.} \frac{\psi_{xy}(\tau_1)}{\psi_{xy}(\tau_2)}$$

exists, a restriction of the type

$$\Delta f_{max} \times \tau_{max} \leqq \text{const.}$$

is present when cross-spectral density measurements are made. Here Δf_{max} is the *maximum* bandwidth that can be used in cross-spectral density investigations and τ_{max} is the maximum delay time between the signals to be correlated. From the above formulae it is seen that the restrictions imposed upon practical correlation time function measurements and those imposed upon cross-spectral density measurements actually oppose each other.

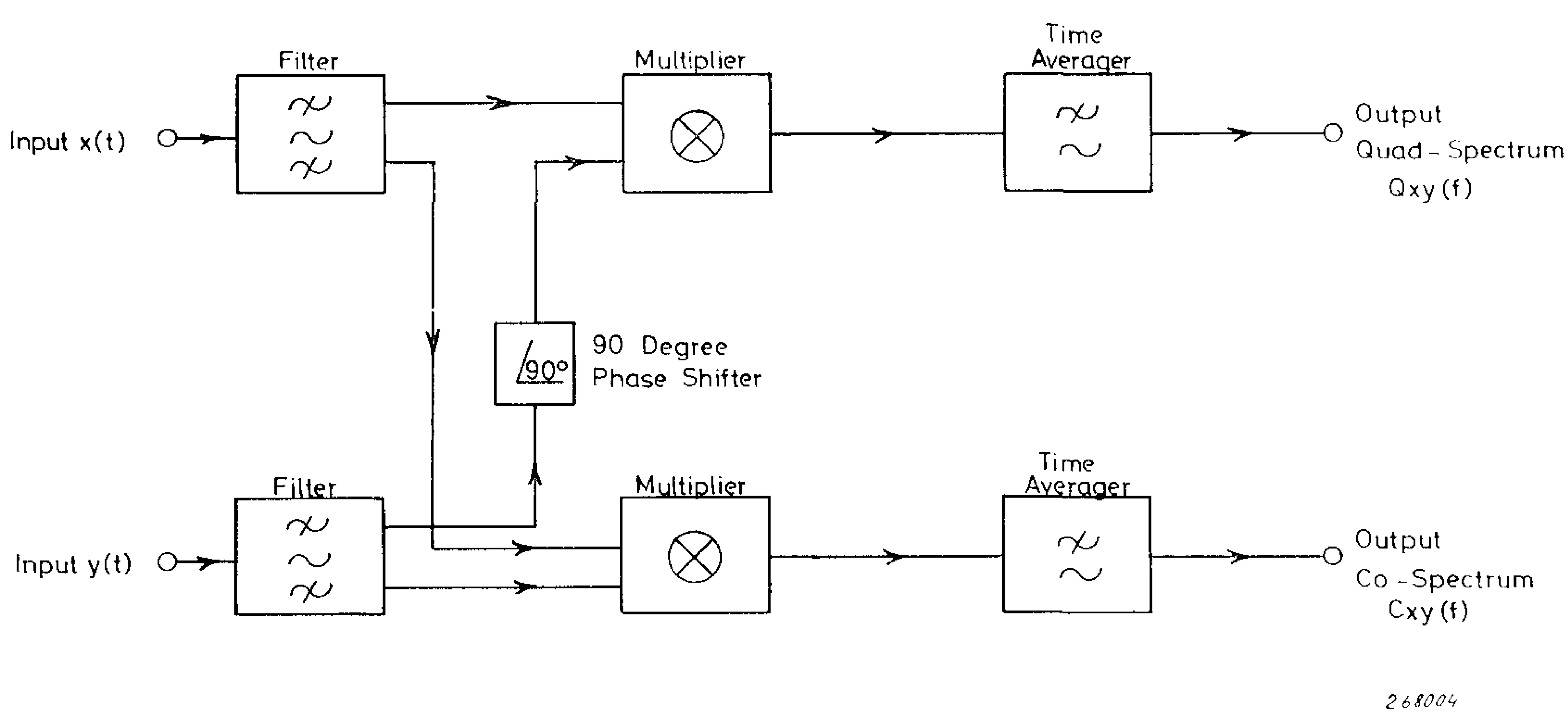


Fig. 9. Principle of operation of an analog cross-spectrum analyzer.

In correlation time function measurements a certain *minimum* measurement bandwidth is required to allow for proper determination of the correlation function maxima, while proper cross-spectral density measurements require the measurement bandwidth to *be smaller* than a value given by the relation constant / τ_{max}.

To obtain an estimate of the value of this constant consider the following:

The mathematical Fourier transform presupposes a continuous frequency analysis with infinitely narrow band filters ($e^{-j2\pi f\tau}$). Such filters do not exist in practice. Commercially available filters have very definite bandwidths and the output of such a filter is therefore also self-correlated (auto-correlated) over very definite time intervals only. These time intervals may be regarded as the "memory" of the filter and it is clear that the multiplications required to obtain the cross-spectral density function must be performed within the "memory time" of the filter. When the "memory" is not perfect the cross-spectral density function obtained will be in error. How large the error will be depends, of course, upon the "memory" of the filters and the time delay between the two signals being multiplied.

Theoretically, the auto correlation function for the output of a box-shaped (ideal) narrow band filter of bandwidth Δf is (see also Fig. 10):

$$\psi(\tau) \approx c\,\Delta f \frac{\sin(\pi\,\Delta f\,\tau)}{\pi\,\Delta f\,\tau} \cos(2\,\pi\,f_0\,\tau)$$

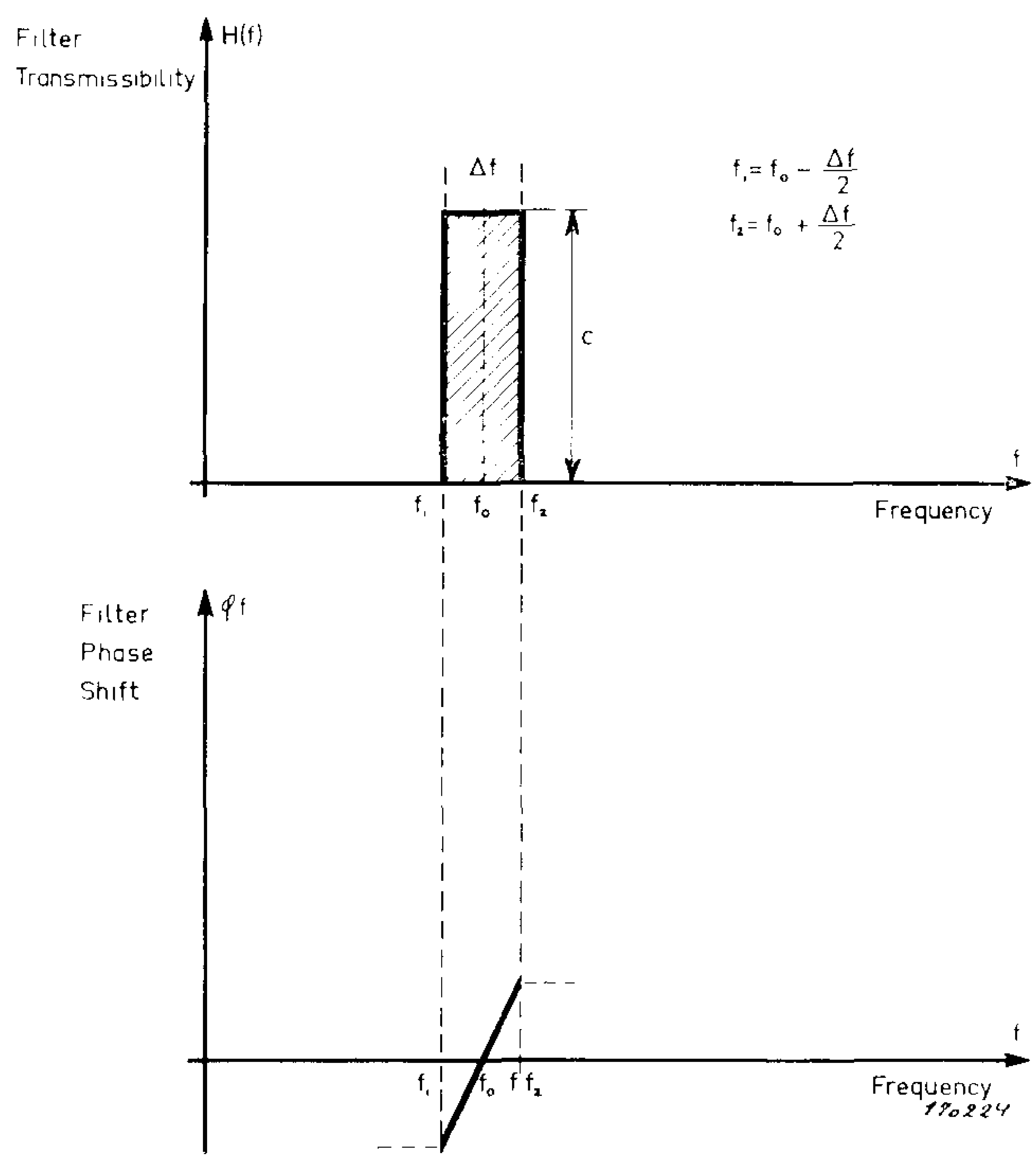


Fig. 10. Sketch illustrating the frequency and phase response of an "ideal" filter, with a maximum response c and bandwidth Δf.

This function is plotted in Fig. 11 for the case when $c\,\Delta f = 1$ and $x = \pi\,\Delta f\,\tau$ and it is seen that as long as $\Delta f\,\tau$ is small no significant loss of "memory" will occur. The practical conclusion which can be drawn from the above equation and Fig. 11 is that the longer the delay time τ between the inputs to the two filters used for cross-spectrum measurements is, the narrower must the bandwidth of the filters be to achieve correct results. For instance, for the cross-spectral density measurements to be correct to within some 10 % the following relation is obtained from Fig. 11:

$$\underline{\Delta f_{max}\, \tau_{max} \leqq 0.3}$$

From the above formulae it is seen that if a certain frequency resolution is required for a particular cross-spectral density measurement the time delay involved must not exceed the "limit"

$$\tau_{max} \leqq \frac{0.3}{\Delta f}$$

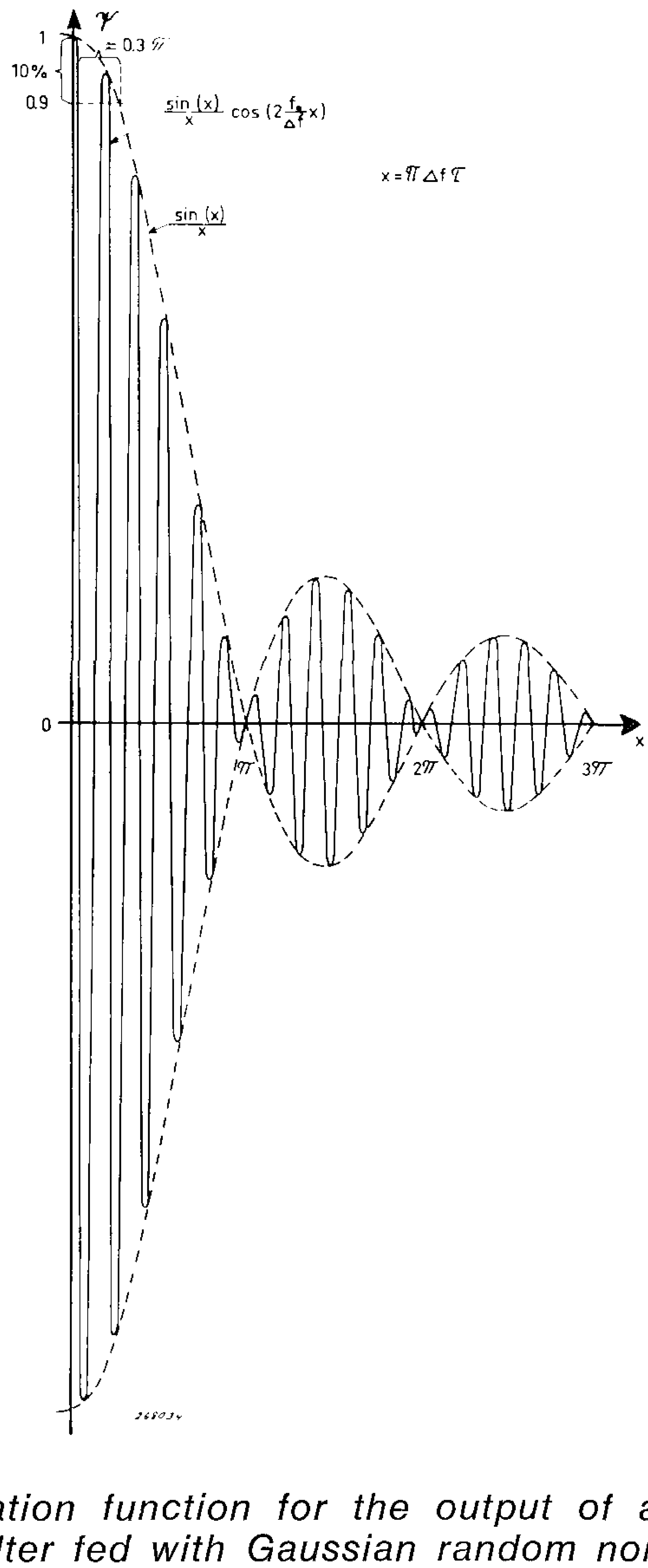


Fig. 11. Auto-correlation function for the output of a narrow band "ideal" filter fed with Gaussian random noise.

Thus if a measurement bandwidth of say 30 Hz is used the measured cross-spectral density value is only "correct" if the maximum time delays in the system is *less* than 0.3/30 = 0,01 s = 10 ms. This requirement makes the technique relatively unattractive for acoustic measurement purposes. *It does,*

on the other hand, make it a very useful tool in the study of structural mechanical vibrations. Here the time delays involved are, as mentioned earlier, often considerably smaller than 10 ms, and frequency resolutions better than 30 Hz are often required. Even if very lightly damped resonances (and thus long delay times) are included in the transmission paths the requirement $\Delta f_{max} \times \tau_{max} \leq 0.3$ is normally fulfilled, as the frequency analysis of such systems require the use of extremely narrow band filters (Δf_{max} very small) to obtain proper measurement resolution.
One of the most interesting applications of the cross-spectral density technique in the field of mechanical vibration studies might be the possibility it offers *to determine complex transfer characteristics in a system without interfering with the system's normal operation.* This kind af measurements is of particular importance in the fields of shipboard, aircraft and space vehicle vibration, but has also been utilized in vibration studies on automobiles and special machinery.

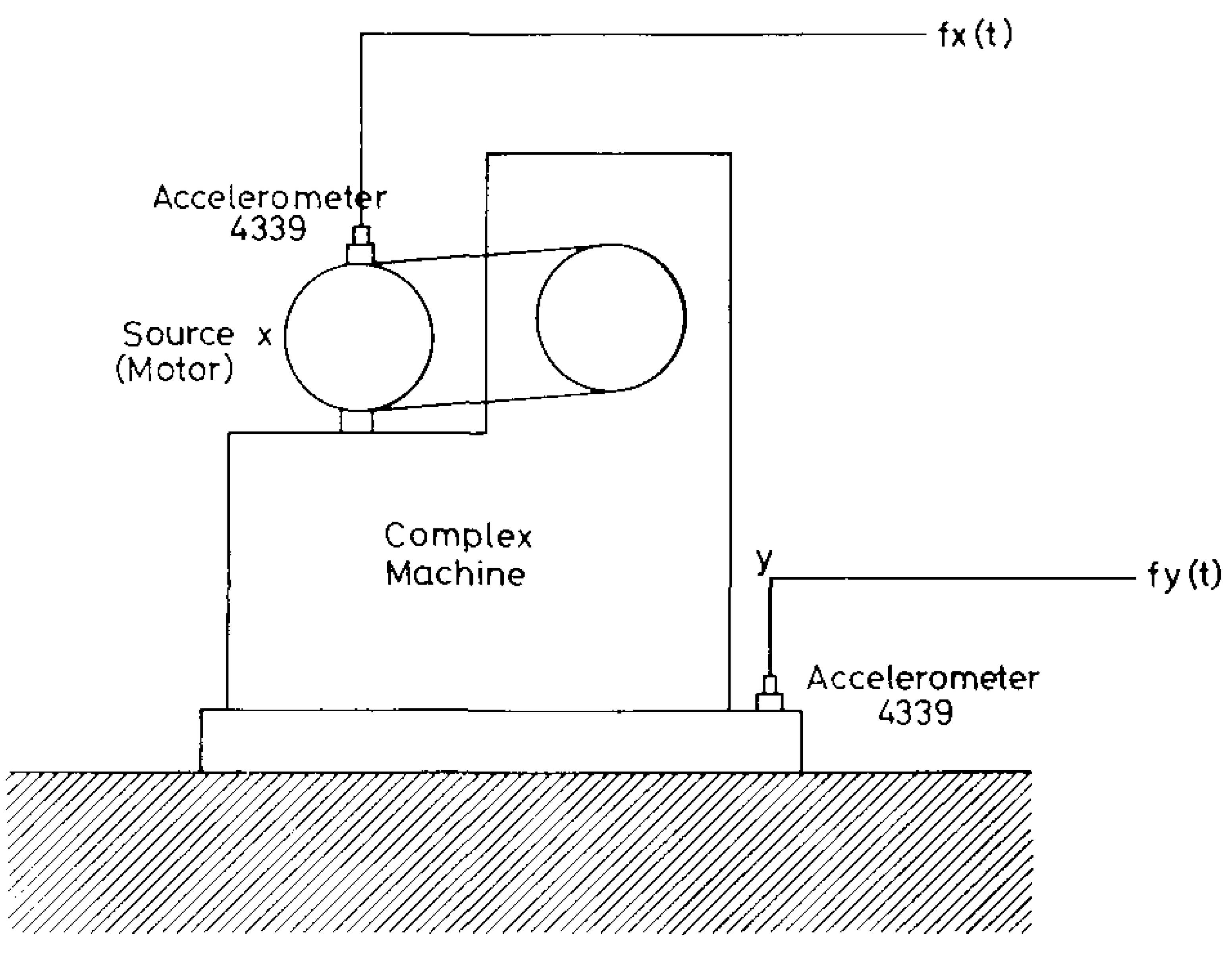


Fig. 12. Illustration of transfer characteristic measurements on a complex machine without interfering with the machine's normal operation.

The relation between the cross-spectral density measured between the point x and the point y in the system sketched in Fig. 12, the ordinary "power" spectral density, $W_{xx}(f)$, measured at x and the (complex) transfer characteristic between x and y, $H_{xy}(f)$ is:

$$\underline{W_{xy}(f) = H_{xy}(f)\, W_{xy}(f)}$$

As

$$H_{xy}(f) = |H_{xy}(f)|\, e^{-j\varphi_{xy}(f)}$$

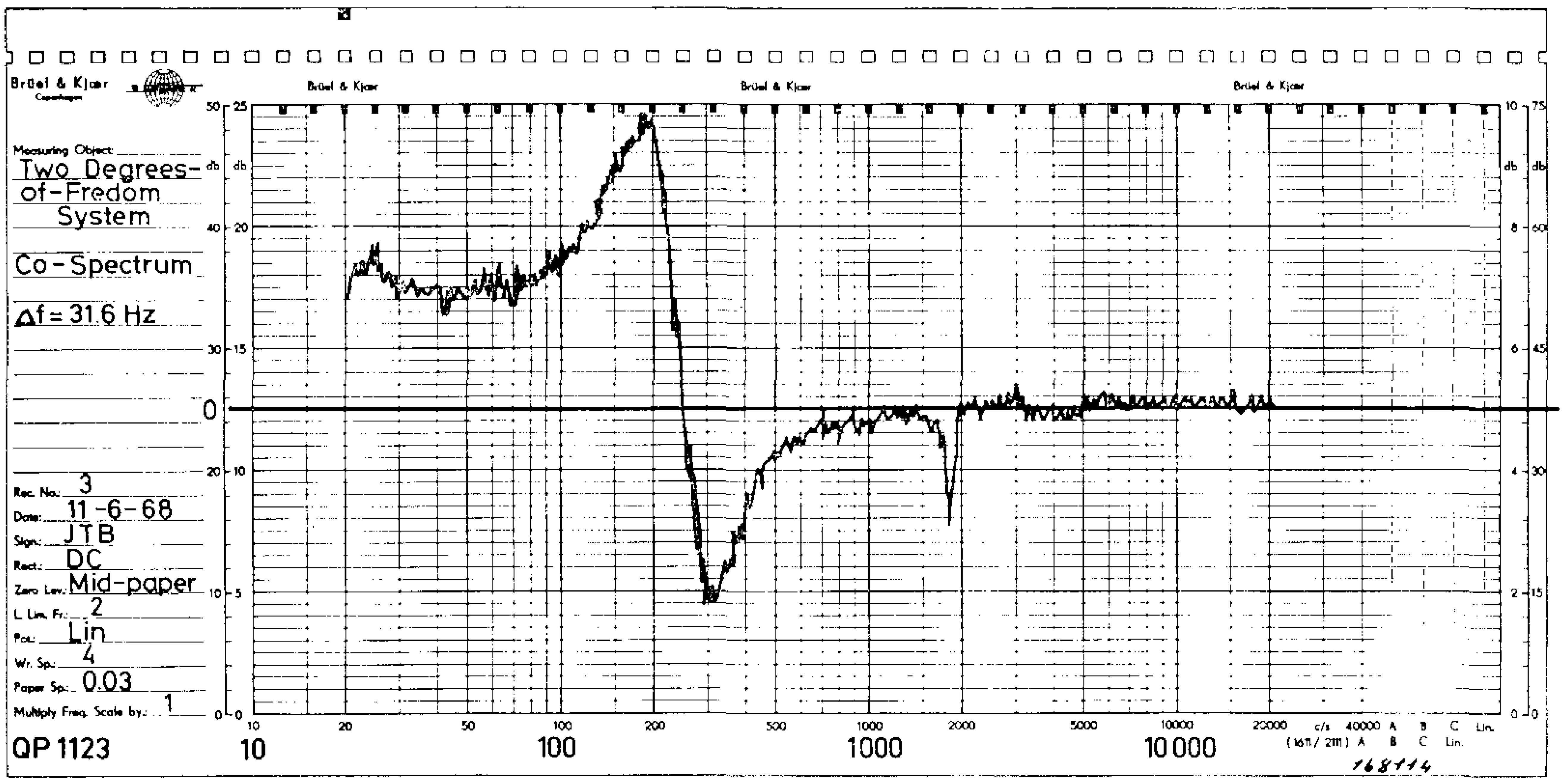
and

$$|W_{xy}(f)| = \sqrt{C^2_{xy}(f) + Q^2_{xy}(f)}$$

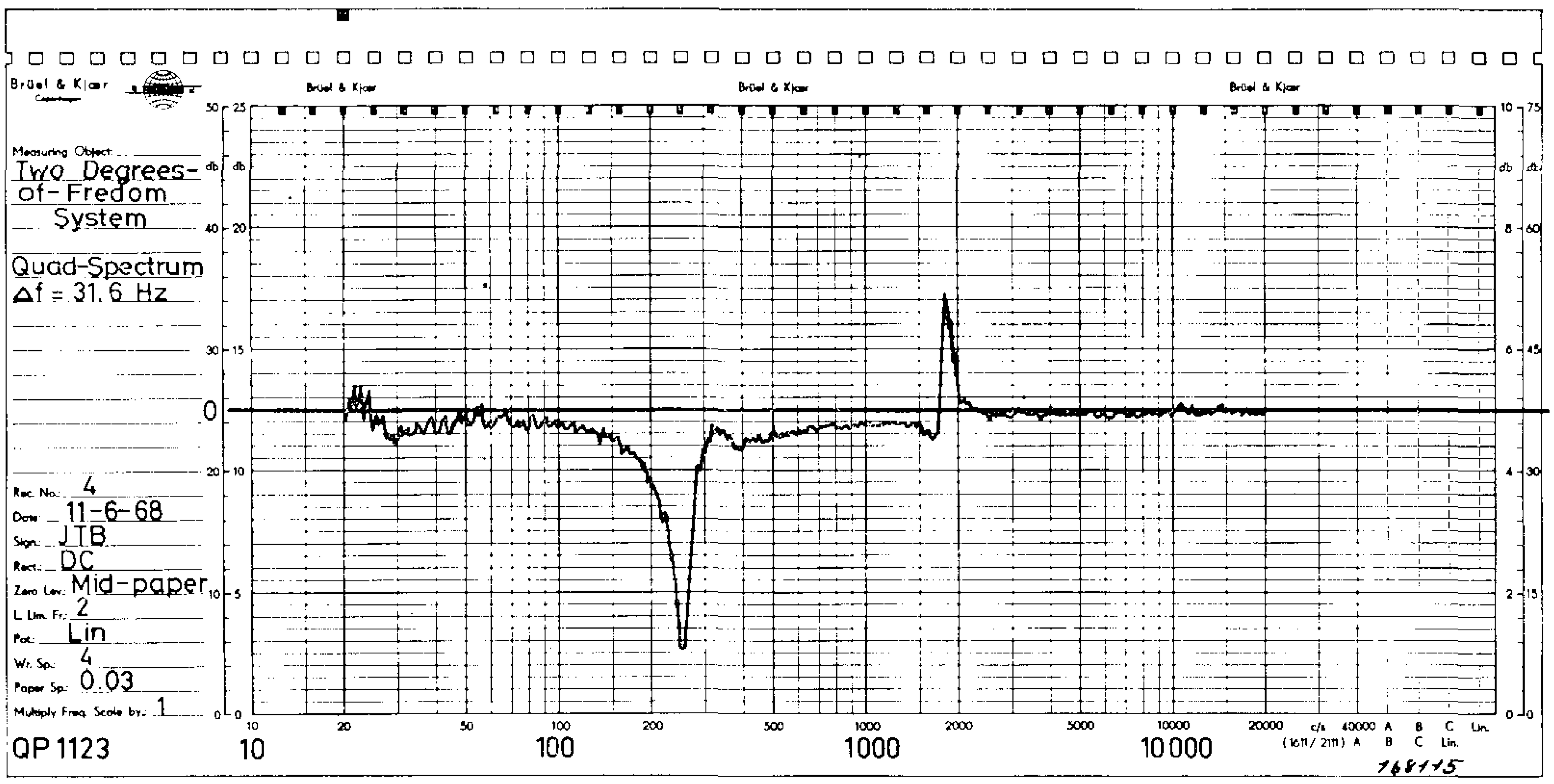
as well as:

$$\varphi_{xy}(f) = \tan^{-1}\left[\frac{Q_{xy}(f)}{C_{xy}(f)}\right]$$

the function $H_{xy}(f)$ can be completely determined from measurements of the co- and quad spectral density functions between x and y, and the "power" spectral density function at x.
To demonstrate the use of this technique a simple experiment has been made at Brüel & Kjær on an electrical analog model consisting of a two degrees-of-freedom system. The resulting co- and quad spectral density

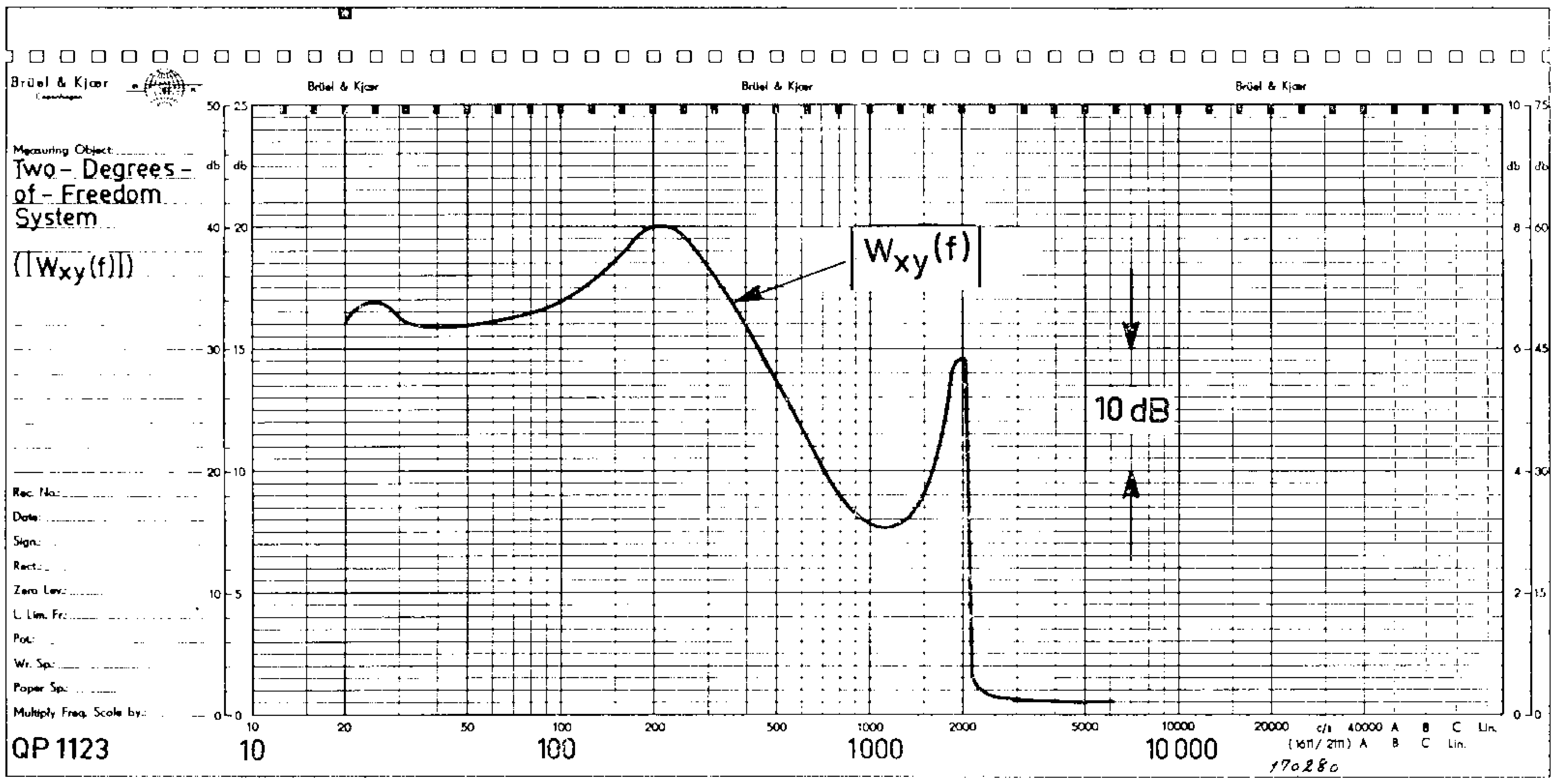


a)

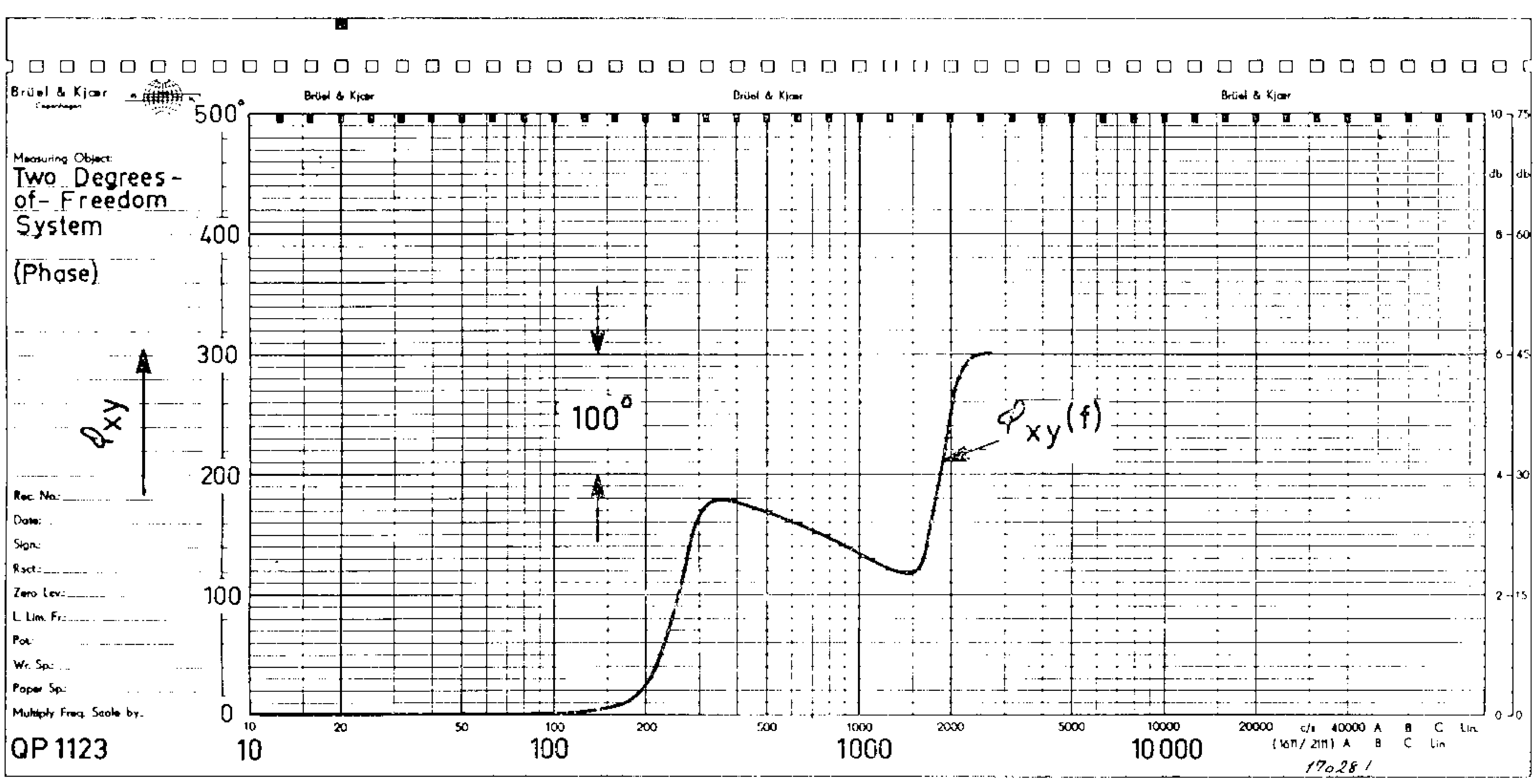


b)

Fig. 13. Cross-spectral density curves relating the output to the input of a two degrees-of-freedom system excited by "white" random noise.
a) Co-spectral density curve.
b) Quad spectral density curve.



a)



b)

Fig. 14. The same measurement results as given in Fig. 13 but this time plotted in terms af $|W_{xy}(f)|$ *and* $\varphi_{xy}(f)$.
a) Modulus of the cross-spectral density function $|W_{xy}(f)|$.
b) Phase shift of the cross-spectral density function $|\varphi_{xy}(f)|$.

functions are shown in Fig. 13, while $|W_{xy}(f)|$ and $\varphi_{xy}(f)$ are plotted in Fig. 14. Because the input to the system in this case consisted of random noise with constant "power" spectral density ($W_{xx}(f) = \text{const.}$) the graphs shown in Fig. 14 at the same time represent $H_{xy}(f)$.

Another interesting function which can be derived from measurements of $W_{xy}(f)$, $W_{xx}(f)$ and $W_{yy}(f)$, Fig. 15, is the so-called *coherence function* $\gamma_{xy}(f)$:

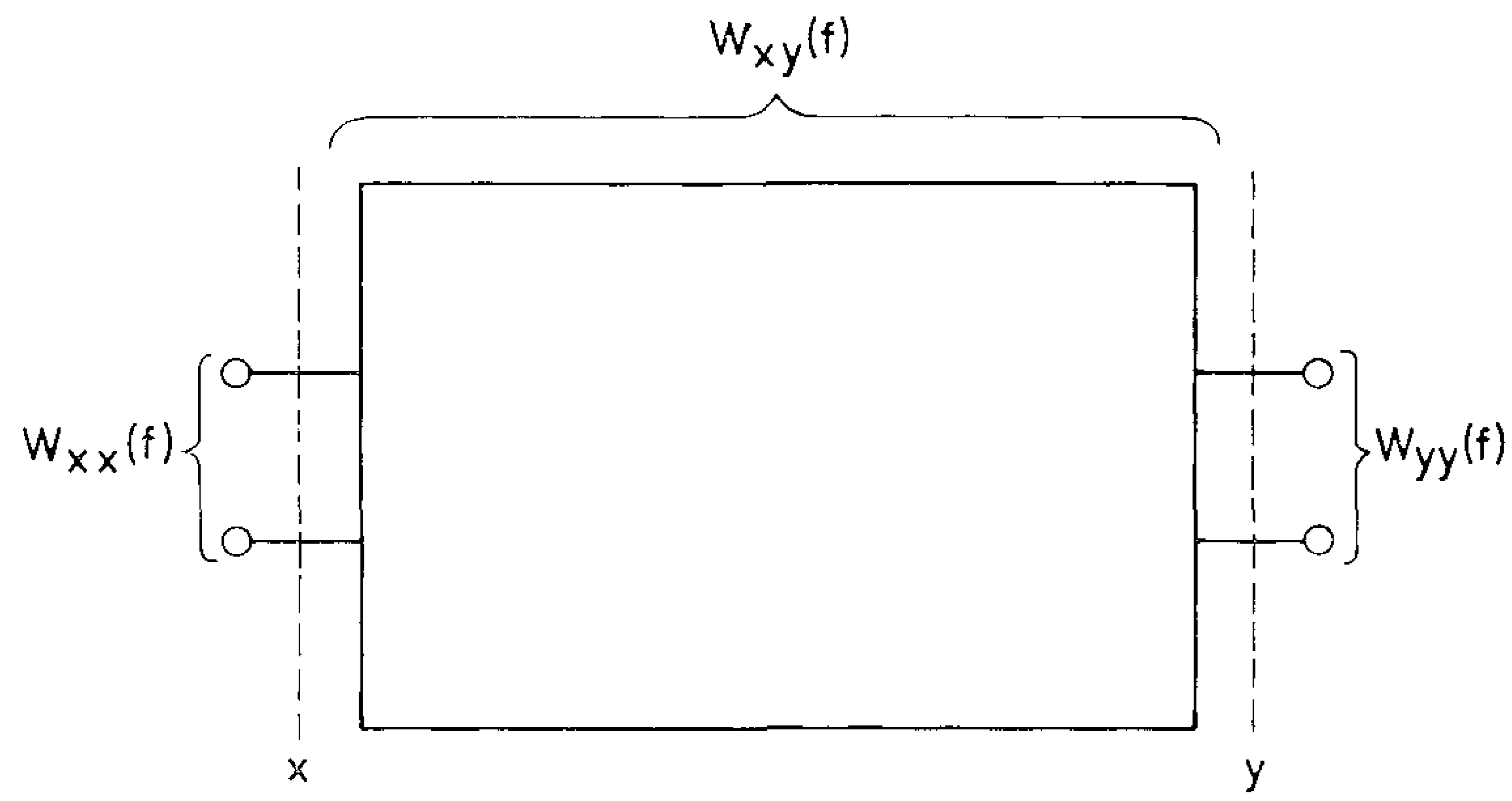


Fig. 15. Sketch illustrating the terms $W_{xx}(f)$, $W_{xy}(f)$ and $W_{yy}(f)$ necessary to determine the coherence function.

$$\gamma_{xy}(f) = \sqrt{\frac{W_{xx}(f)\,W_{yy}(f)}{|W_{xy}(f)|^2}}$$

The coherence function actually is a measure of the maximum (time independent) correlation between the signals present at x and y. It is, however, *frequency dependent,* and is not to be confused with the (time delay dependent) correlation time function.

Conclusion

It was mentioned in the introduction that the cross-correlation function and the cross-spectral density function constitute a Fourier transform pair, and that they therefore contain the same amount of information. Also, it was stated that which of the two functions should be used to solve a particular practical problem was thus basically a matter of convenience. The discussion carried out in this paper leads, however, to the conclusion that in most practical vibration engineering cases a certain preference must be given to the cross-spectral density representation of the information. This is due to the frequency dependency present in most of the mechanical systems found in practice.

Looking at the problem from the point of view of interpreting graphically recorded curves it might, furthermore, be useful to bear the following brief discussion in mind:

A curve of the type shown in Fig. 16 is normally not easily interpreted, looking somewhat like random vibrations. By applying Fourier analysis techniques to the signal a curve of the type shown in Fig. 17 is obtained. This curve is, from an engineering point of view, considerably easier to interprete than that given in Fig. 16. In the case shown it is more or less obvious that the most important regions of the curve, Fig. 17, are the regions around the maxima.

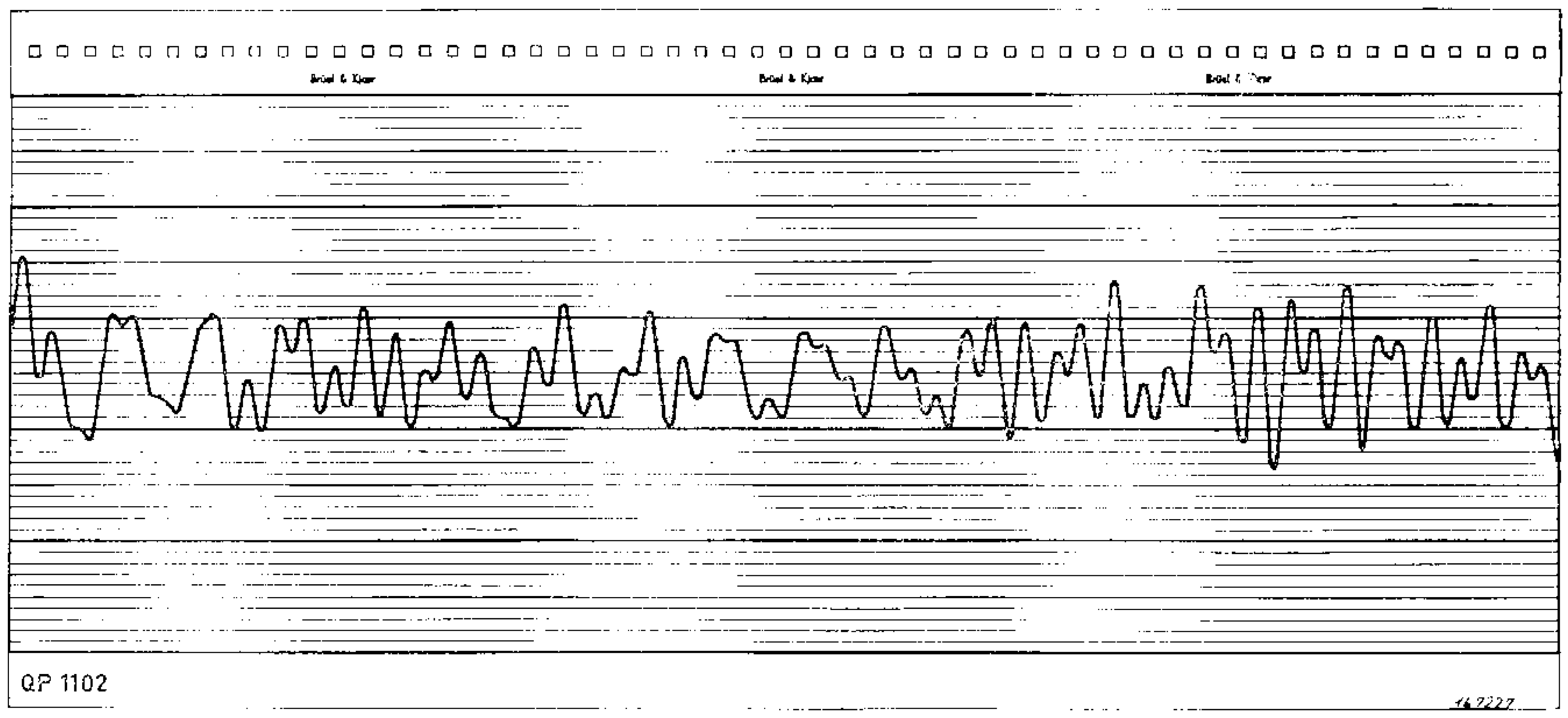


Fig. 16. Time function of the response of a randomly excited multi-resonant system.

Because the curve represents a magnitude-versus-frequency diagram it suggests that here three predominant frequency regions (randomly excited resonances) are present, a fact which is not very obvious from a direct examination of the magnitude-versus-time curve given in Fig. 16.
Similarly, a curve of the type shown in Fig. 18 (see also Fig. 5) immediately suggests that two regions in time (time delays) seem to be important. The modulus of the inverse Fourier transform of the curve, Fig. 18, looks like the graph sketched in Fig. 19.
Thus, generally speaking, the use of Fourier analysis methods convert curves of the type shown in Figs. 16 and 19 to curves of the type in Figs. 17 and 18. The major advantage obtained by applying Fourier analysis techniques to curves like those given in Figs. 16 and 19, is consequently, that a more readily interpretable data presentation is achieved.

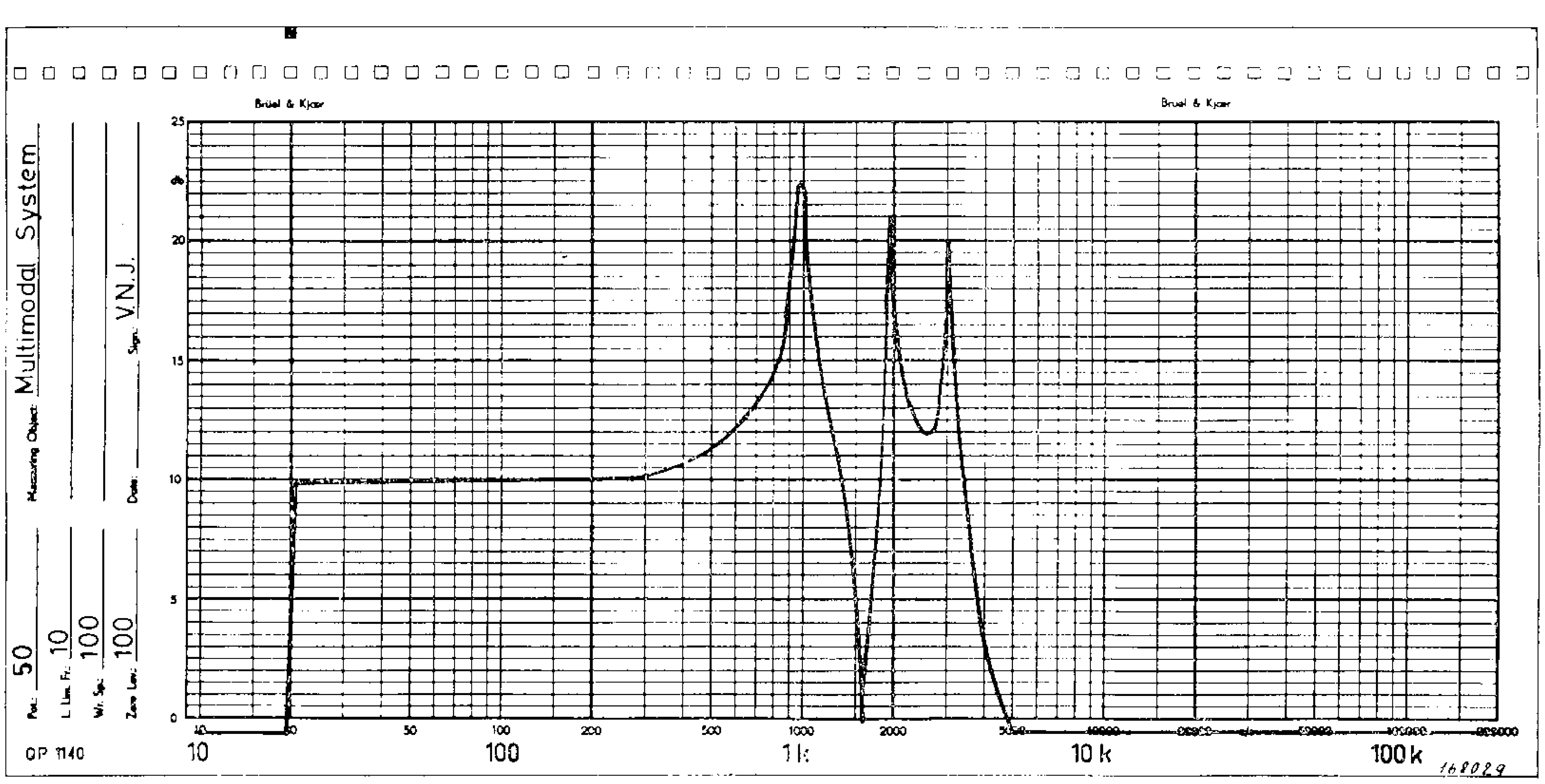


Fig. 17. Fourier analysis of the time function shown in Fig. 16.

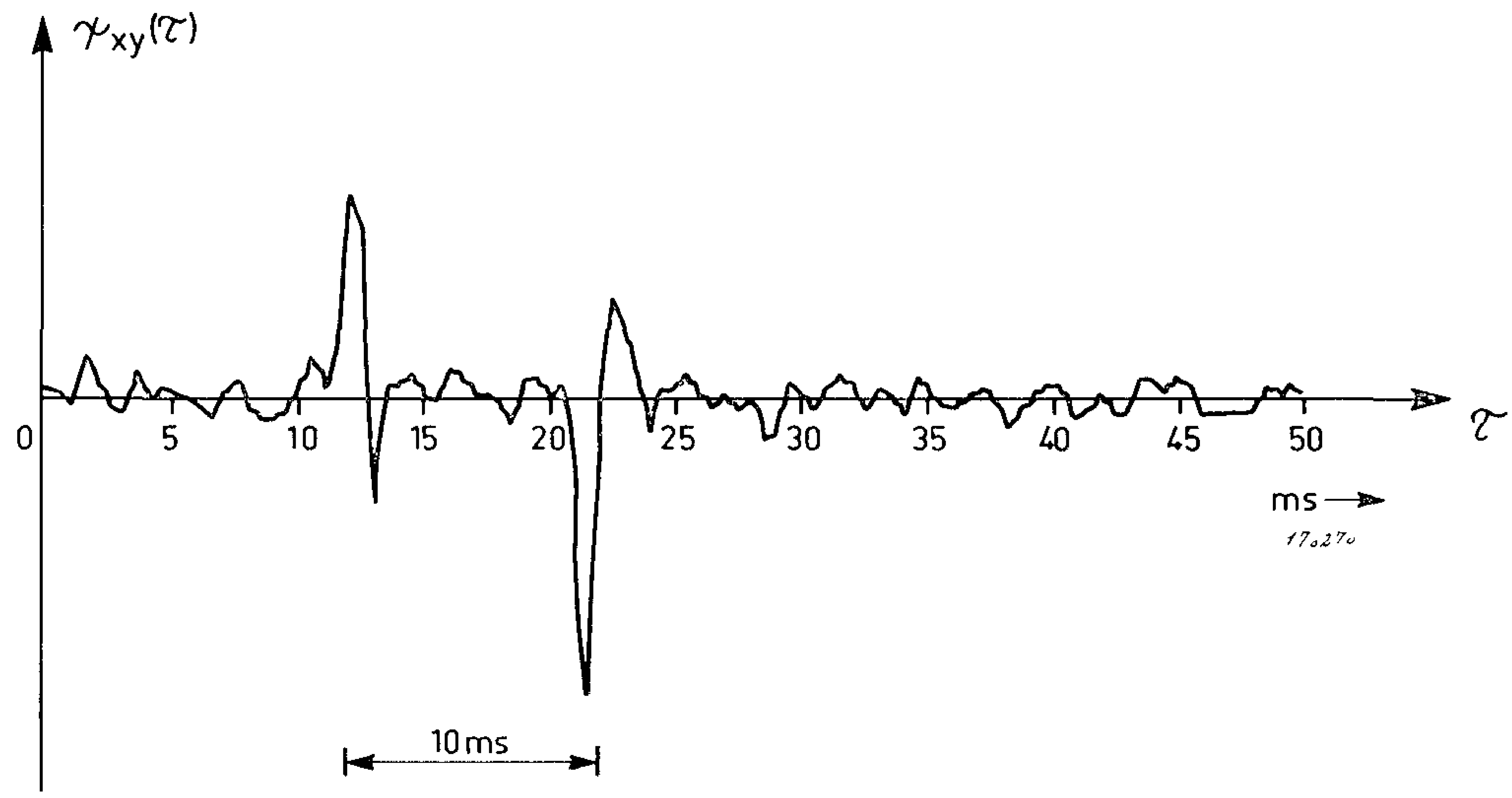


Fig. 18. Cross-correlation time function of a wide band system with two distinct time delays.

From the similarity of the two Fourier integrals:

$$W_{xy}(f) = \int_{-\infty}^{\infty} \psi_{xy}(\tau)\, e^{-j2\pi f \tau}\, d\tau$$

and

$$\psi_{xy}(\tau) = \int_{-\infty}^{\infty} W_{xy}(f)\, e^{-j2\pi f \tau}\, df$$

it can be seen that when one of the functions looks like the curve plotted

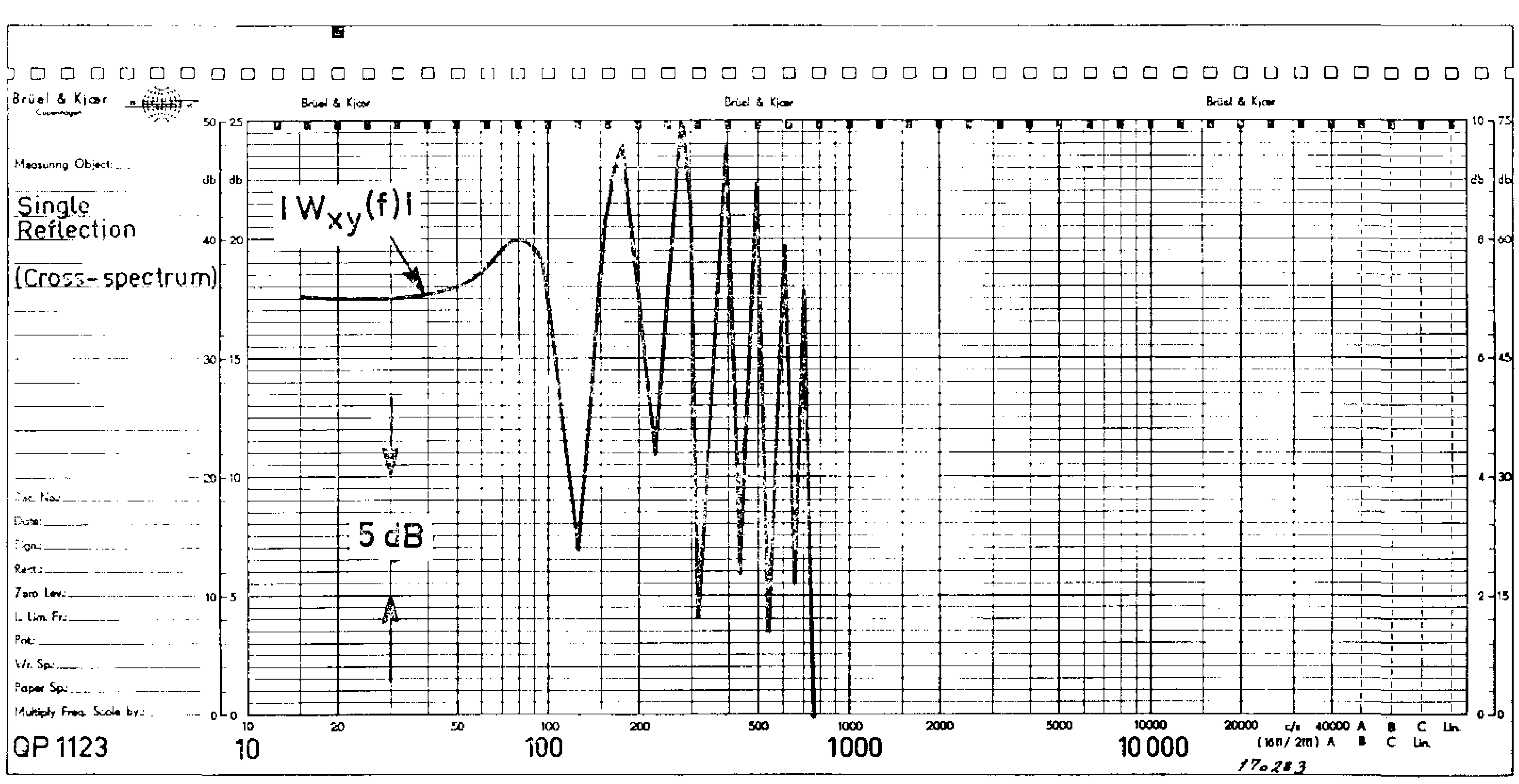


Fig. 19. Fourier analysis of the correlation time function shown in Fig. 18.

in Fig. 16, then the other will turn out to look like that shown in Fig. 17 and vice versa. (Actually, as complex functions are involved one may, in some cases, have to consider both the real and the imaginary parts of the integrals separately. However, in principle, a "wave-like" curve in one domain, gives, by Fourier transformation, a "peaked" curve in the other domain).

Thus, if the system being investigated is strongly frequency dependent (and nearly time independent) the cross-spectral density function will be a "peaked", relatively easily interpretable curve, while the cross-correlation function will have a "wave-like" shape. Similarly, if the system is strongly time dependent (due for instance to reflections), and nearly frequency independent, the cross-correlation function is "peaked" while the cross-spectral density function is normally "wave-like".

The terms "nearly time independent" and "nearly frequency independent" have been discussed to some extent in the preceeding text, and it was found that:

1. Nearly time independent means that the condition

$$\Delta f_{max}\, \tau_{max} \leqq 0.3$$

   must be fulfilled. Cross-spectral density data can then be readily interpreted, and

2. "Nearly frequency independent" means fulfillment of the condition

$$\Delta f_{min}\, \tau_s \geqq 3\, \frac{\psi_{xy}(\tau_1)}{\psi_x(\tau_2)}$$

   in this case cross-correlation time function data are readily interpretable.

Systems which do not fulfil one or the other of these conditions can not easily be analyzed neither by cross-correlation function techniques nor by cross-spectral density function techniques.

Although the techniques discussed in this paper constitute powerful research tools in many fields of technical importance it might be adviceable, before any elaborate measurement programme is laid down, to consider the "basic" limitations pointed out here.

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Appendix A

Derivation and Discussion of the Criterion

$$\Delta f_{min}\,\tau_s \geq 3\,\frac{\psi_{xy}(\tau_1)}{\psi_{xy}(\tau_2)}.$$

As shown in Appendix B the cross-correlation function between the input and output of a system excited by stationary, "white" random noise is equal to a constant times the system's unit impulse response:

$$\psi_{xy}(\tau) = \text{constant} \times h(\tau)$$

Considering now an ideal filter, Fig. A.1, the unit impulse response of such a filter is*):

$$h(\tau) = 2\,\Delta f\,\frac{\sin\left[\pi\,\Delta f\,(\tau-\tau_L)\right]}{\pi\,\Delta f\,(\tau-\tau_L)}\cos(2\,\pi\,f_o\,\tau).$$

*) See for instance: J. T. Broch and H. P. Olesen "On the Frequency Analysis of Mechanical Shocks and Single Impulses". Brüel & Kjær Techn. Rev. No. 3 — 1970.

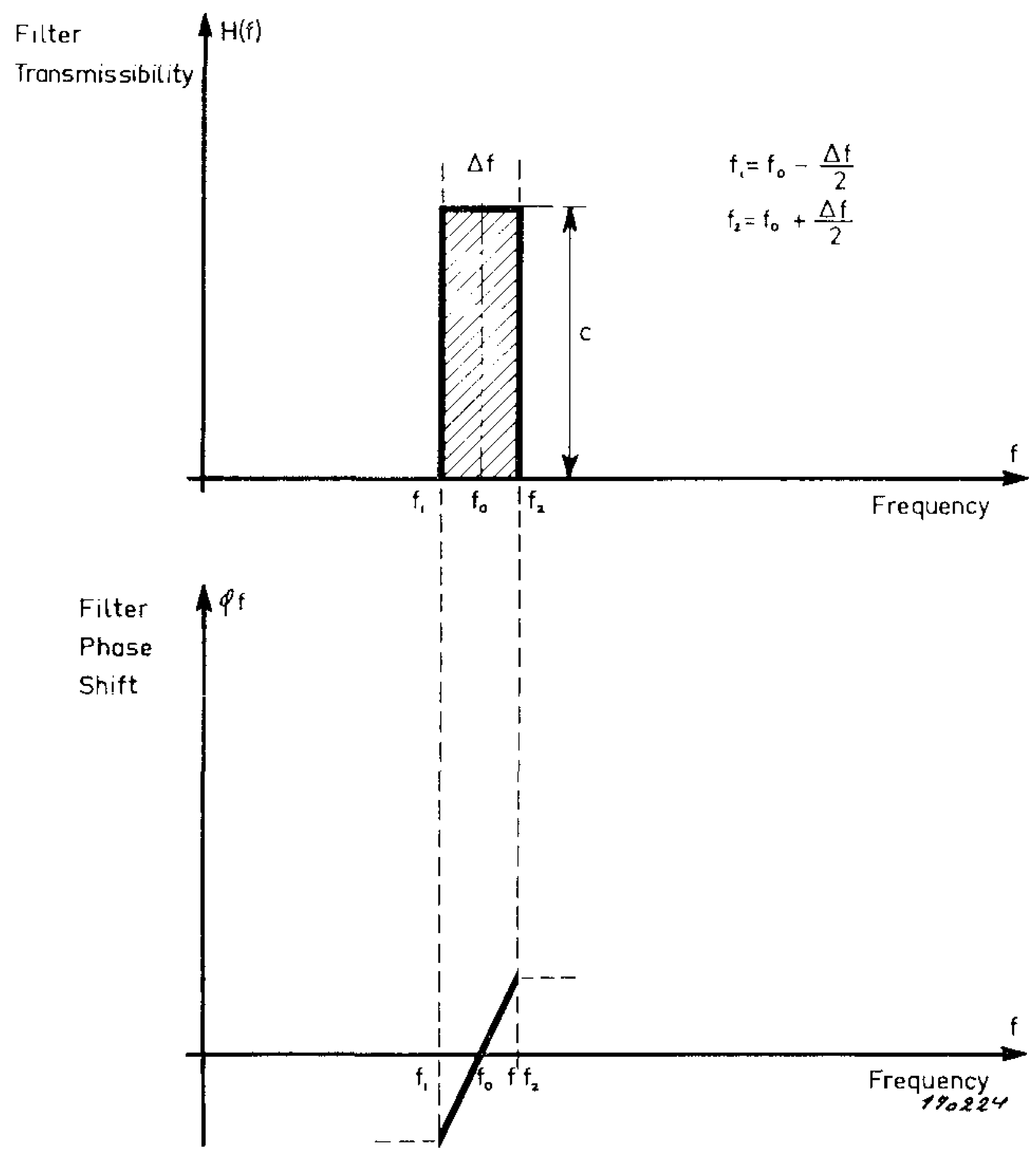


Fig. A.1. Sketch illustrating the frequency and phase response of an "ideal' filter.

This function is sketched in Fig. A.2, and represents, according to the above, also the cross-correlation function for the direct sound path, Fig. A.3, with $\tau_L = \tau_1$.

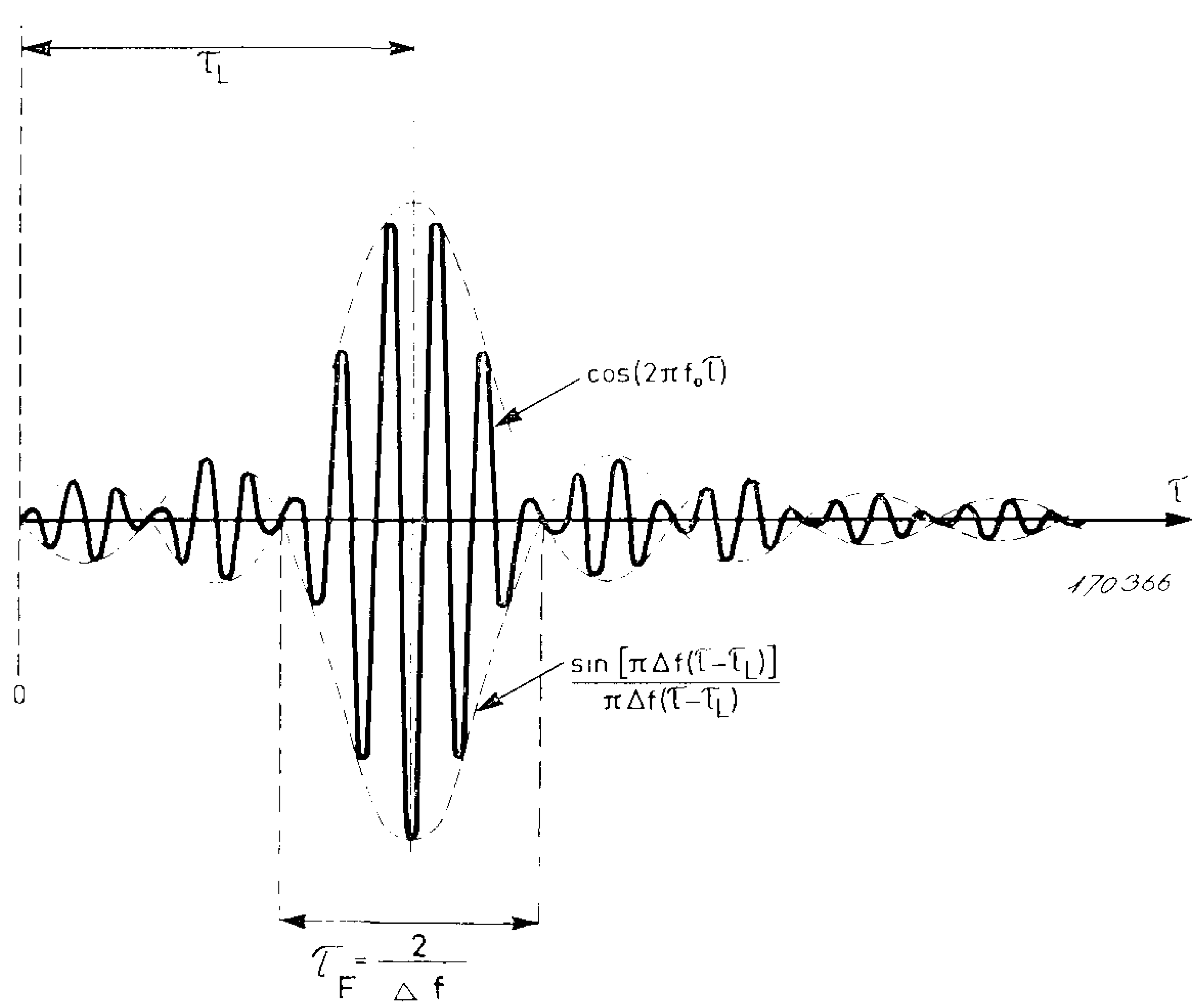


Fig. A.2. Sketch illustrating the unit impulse response of an "ideal" filter.

If a second sound path exists, caused for instance by reflection, and the time, τ_2, taken for the sound to travel from the loudspeaker (Microphone No. 1), via the reflecting surface, to Microphone No. 2 is very much larger than τ_1 a correlogram of the type shown in Fig. A.4 is obtained.

When τ_2 is not very much larger than τ_1 the resulting correlogram will be a superposition of the two responses shown in Fig. A.4, and errors in the measured magnitude of the correlations occur. Similarly, if the bandwidth of the filters used is decreased the widths of the major correlation "lobes" $\frac{2}{\Delta f}$, see Fig. A.2) is increased, and again errors will occur in the measurements.

To obtain an estimate of the errors involved consider Fig. A.4 where it can

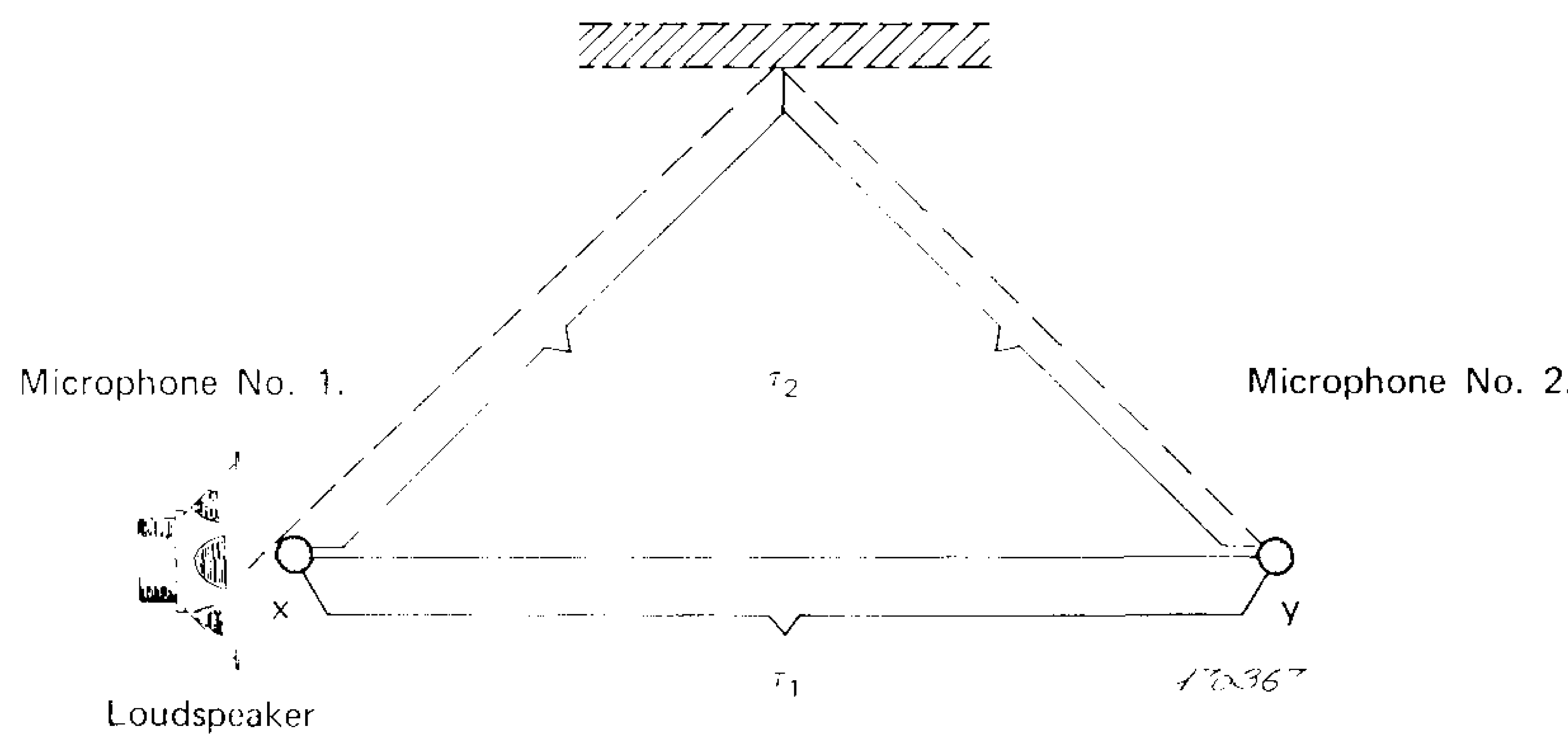


Fig. A.3. Illustration of a system containing two distinct time delays (see also Fig. 3 of the text).

be seen that the overall envelope of the first correlation maximum decreases with τ and Δf as

$$\psi_1(\tau) \rightarrow \frac{\text{const.}}{\pi \Delta f (\tau - \tau_1)}$$

i.e. according to a hyperbola.

Disregarding phase relationships (and considering the maximum errors only) then the measured value of the second correlation maximum $\psi^1(\tau_2)$, is:

$$\psi^1(\tau_2) = \psi(\tau_2) + \psi_1(\tau_2)$$

Here $\psi(\tau_2)$ is the correct value for the second correlation maximum and $\psi_1(\tau_2)$ is the maximum value of the first correlation function, $\psi_1(\tau)$, at the time τ_2. To determine the maximum value of $\psi_1(\tau)$ at the time τ_2 the constant in the expression (hyperbola):

$$\psi_1(\tau) = \frac{\text{const.}}{\pi \Delta f (\tau - \tau_1)}$$

must be determined.

By studying Fig. A.4 it is seen that when $\pi \Delta f (\tau - \tau_1) \approx 3\pi$ then $\psi_1(\tau) \approx 0.1\,\psi(\tau_1)$.

Thus:

$$\psi_1(\tau) = 0.1\,\psi(\tau_1) = \frac{\text{const.}}{3\pi}$$

i.e.

$$\text{const.} = 0.3\,\pi\,\psi(\tau_1)$$

and

$$\psi_1(\tau) = \frac{0.3\,\pi\,\psi(\tau_1)}{\pi \Delta f (\tau - \tau_1)} = \frac{0.3\,\psi(\tau_1)}{\Delta f (\tau - \tau_1)}$$

At the time τ_2 then:

$$\psi_1(\tau_2) = \frac{0.3\,\psi(\tau_1)}{\Delta f (\tau_2 - \tau_1)}$$

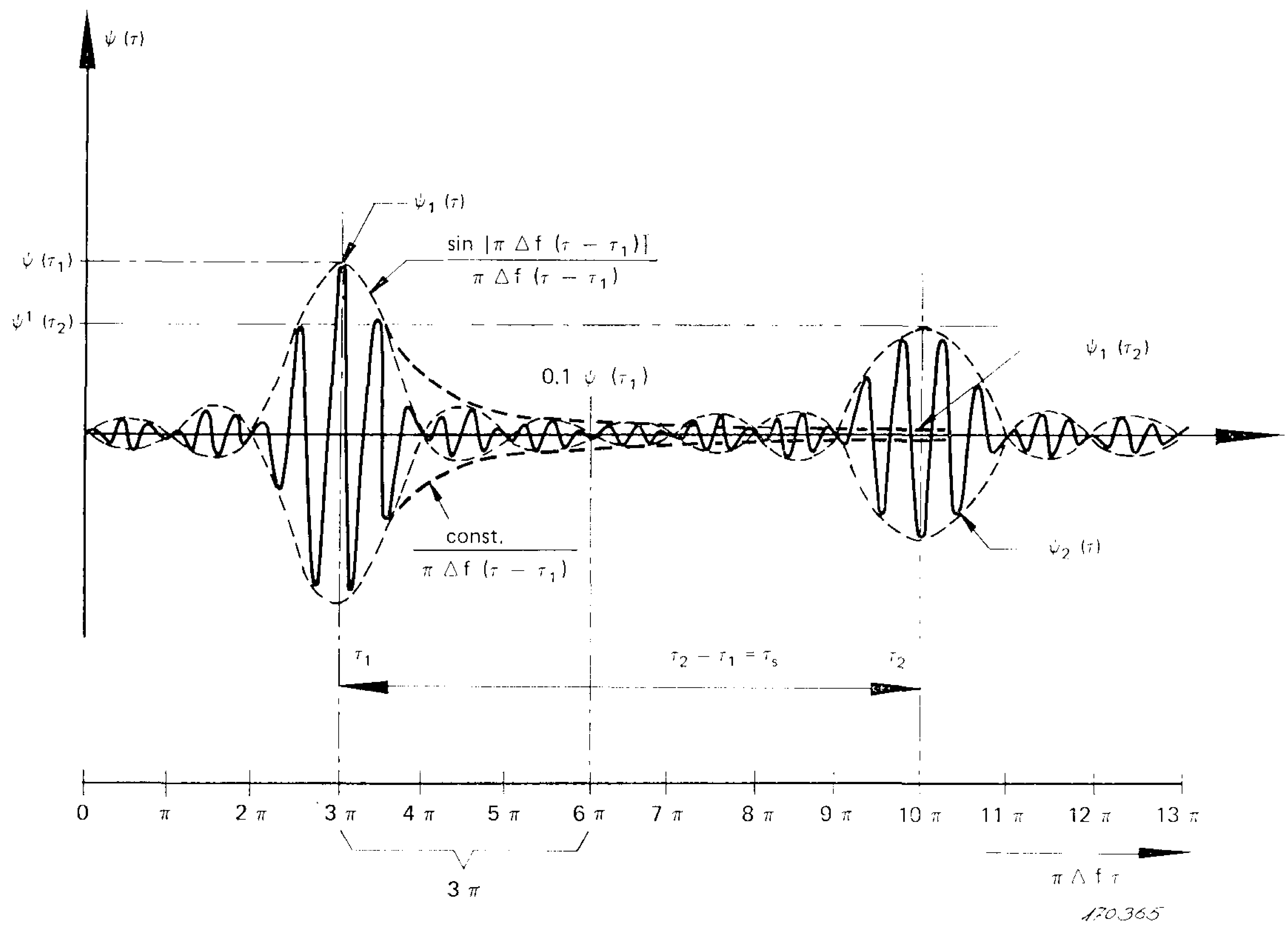


Fig. A.4. Theoretical correlogram obtained when measurements are made on a system containing two distinct, and widely separated, time delays. The use of "ideal" filters is assumed, and the theoretical cross-correlation functions and envelopes obtained are indicated.

Hereby the measured value of $\psi(\tau_2)$ at τ_2 is:

$$\psi^1(\tau_2) = \psi(\tau_2) + \frac{0.3\,\psi(\tau_1)}{\Delta f (\tau_2 - \tau_1)}$$

And the relative measurement error:

$$\frac{\psi^1(\tau_2) - \psi(\tau_2)}{\psi(\tau_2)} = \frac{0.3}{\Delta f (\tau_2 - \tau_1)} \; \frac{\psi(\tau_1)}{\psi(\tau_2)}$$

If a relative error of 10 % is acceptable, i.e.

$$\frac{\psi^1(\tau_2) - \psi(\tau_2)}{\psi(\tau_2)} = 0.1$$

then

$$0.1 = \frac{0.3}{\Delta f\,(\tau_2 - \tau_1)}\;\frac{\psi(\tau_1)}{\psi(\tau_2)}$$

or

$$\Delta f\,(\tau_2 - \tau_1) = \Delta f\,\tau_s = 3\,\frac{\psi(\tau_1)}{\psi(\tau_2)}$$

where

$$\tau_2 - \tau_1 = \tau_s$$

This is the expression given in the text. It is based on the use of ideal filters and a *maximum* acceptable magnitude error of 10 %.

Practical filters do not respond exactly according to the function

$$h(\tau) = 2\,\Delta f\,\frac{\sin[\pi\,\Delta f\,(\tau - \tau_L)]}{\pi\,\Delta f\,(\tau - \tau_L)}\,\cos(2\,\pi\,f_o\,\tau)$$

when exposed to a unit impulse. The envelope function decreases much more rapidly than a $\frac{\sin(x)}{x}$ function, in general exponentially, and the factor

$$3\,\frac{\psi(\tau_1)}{\psi(\tau_2)}$$

may therefore in practical cases be reduced to say

$$2\,\frac{\psi(\tau_1)}{\psi(\tau_2)}$$

(or even less) depending upon the desired measurement accuracy. It does, however, nevertheless set a rather strict limit to the obtainable frequency resolution in a correlation time function measurement.

Appendix B

Relationship Between a System's Unit Impulse Response and the Cross Correlation Function Between Input and Output

If the input to a frequency limited system as sketched in Fig. B.1 is connected to a wide-band random noise source with constant power spectral density, $W_{xx}(f)$, the cross spectral density function between the system's output and input is given by the expression:

$$W_{xy}(f) = H_{xy}(f)\,W_{xx}(f) = H_{xy}(f)\;x\;\text{constant}$$

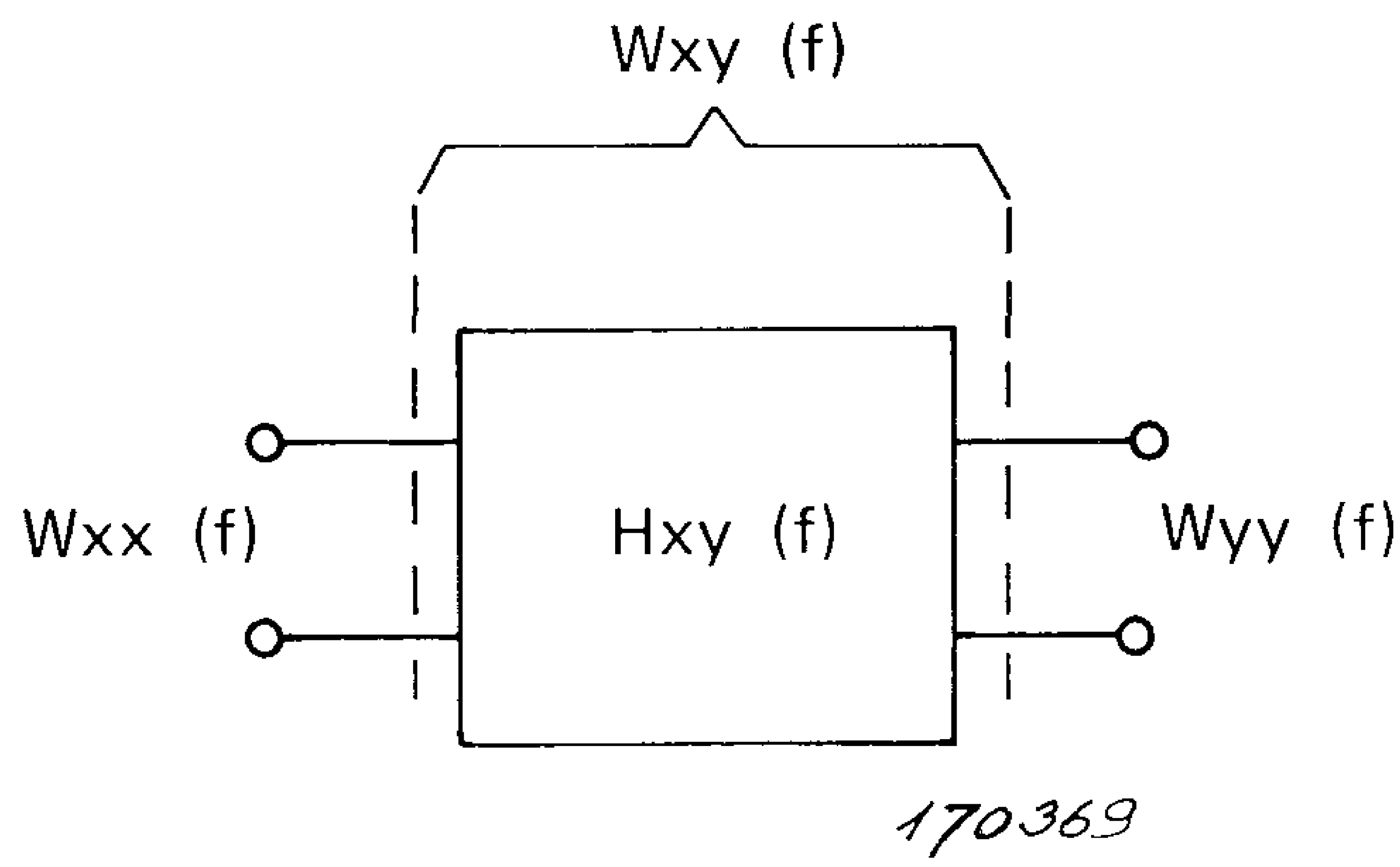


Fig. B.1. Illustration of a "black box" system. The system is assumed to be frequency limited so that the bandwidth of the input signal, $W_{xx}(f)$, is considerably wider than that represented by the transfer function of the "black box".

where

$W_{xy}(f)$ = Cross Spectral Density (complex)
$H_{xy}(f)$ = System Frequency Response Function (complex)
$W_{xx}(f)$ = Input Power Spectral Density = constant

By taking the inverse Fourier transform of both sides of the above equation one obtains:

$$\int_{-\infty}^{\infty} W_{xy}(f)\, e^{j2\pi f\tau}\, df = \text{constant} \times \int_{-\infty}^{\infty} H_{xy}(f)\, e^{j2\pi f\tau}\, df$$

By definition:

$$\int_{-\infty}^{\infty} W_{xy}(f)\, e^{j2\pi f\tau}\, df = \psi_{xy}(\tau)$$

Also the inverse Fourier transform of a system's frequency response function is its unit impulse response, so that:

$$\int_{-\infty}^{\infty} H_{xy}(f)\, e^{j2\pi f\tau}\, df = h(\tau)$$

whereby

$$\psi_{xy}(\tau) = \text{constant} \times h(\tau)$$

News from the Factory

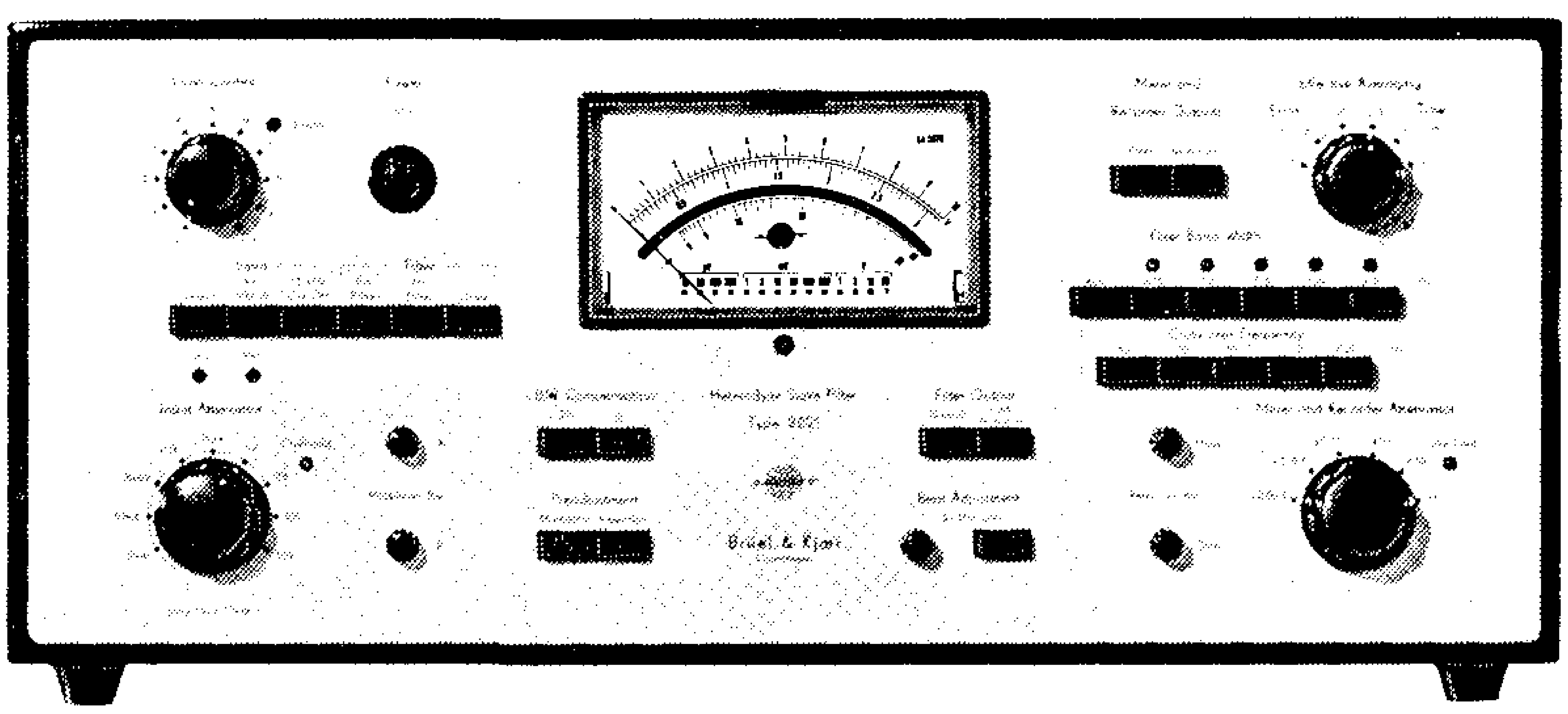
Heterodyne Slave Filter Type 2021

The new Heterodyne Slave Filter is designed for operation with the B & K range of vibration generators. The centre frequency is automatically tuned to the generator frequency in the range 1 Hz to 10 kHz.

Five constant bandwidts from 3.16 Hz to 316 Hz can either be selected manually, or bandwidth changes at predetermined frequencies can be programmed. At the same time $1/\sqrt{B}$ compensation for bandwidth can be included for noise operation.

An automatic bandwidth/averaging time combination can be set in three positions giving $\leq 3\%$, $\leq 10\%$ or $\leq 30\%$ standard deviation of the measured output at all bandwidths.

A built-in preamplifier ensures that a max. signal to noise ratio can be obtained.



The filter is extremely well suited for all types of vibration analysis and for operation in servo loops in vibration test installations. For such applications a unity gain filtered output with 75 dB dynamic range and a unity gain rejection output with 45 dB rejection are provided. The latter facilitates automatic distortion analysis.

Independent of the servo signals, the recorder output socket can supply either an AC signal with a dynamic range of 75 dB for level recorders or a DC signal proportional to the RMS value over a dynamic range of 50 dB suitable for XY recorders.

Furthermore the 60 kHz output of the filter can be used for mechanical impedance measurements and cross-spectrum analysis as the phase difference between two filters driven by the same generator is less than 1°. (For cross-spectrum measurements one of the filter outputs can be changed 90° in phase).

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(Continued on cover page 2)

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